EXERCISE 4.1

Prove the following by using the principle of mathematical induction for all n∈N

1. $1 + 3 + 3^2 + \ldots + 3^{n-1} = \frac{(3^n - 1)}{2}$

Sol. Let P(n) be the given statement.

i.e. $P(n) : 1 + 3 + 3^2 + \ldots + 3^{n-1} = \frac{(3^n - 1)}{2}$

Putting $n = 1$, $P(1) : 1 = \frac{3^1 - 1}{2}$ or $1 = 1$

P(n) is true for $n = 1$

Assume that P(k) is true

$P(k) : 1 + 3 + 3^2 + \ldots + 3^{k-1} = \frac{(3^k - 1)}{2}$ … (i)

We shall prove that P (k + 1) is true whenever P(k) is true.

Adding 3^k to both sides of (i):

$$1 + 3 + 3^2 + \ldots + 3^{k-1} + 3^k = \frac{(3^k - 1)}{2} + 3^k$$

$$= \frac{3^k - 1 + 2.3^k}{2}$$

$$= \frac{(1+2)3^k - 1}{2}$$

$$= \frac{3.3^k - 1}{2}$$

$$= \frac{3^{k+1} - 1}{2}$$

∴ P(k + 1) is also true whenever P(k) is true.

Hence, P(n) is true for all $n \in$ N.

2. $1^3 + 2^3 + 3^3 + \ldots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$

Sol. Let $P(n) = 1^3 + 2^3 + 3^3 + \ldots + n^3 = \frac{n^2(n+1)^2}{4}$ … (i)

for $n = 1$,

LHS $= 1^3 = 1$

and RHS $= \frac{1^2(1+1)^2}{4} = \frac{1\times 4}{4} = 1$

$\therefore$ LHS = RHS, i.e. P(1) is true.

Let us suppose that P(k) be true

$\therefore$ Putting $n = k$ in (i) we have

$$P(k) : 1^3 + 2^3 + 3^3 + \ldots + k^3 = \frac{k^2(k+1)^2}{4} \qquad \ldots(ii)$$

Adding $(k+1)^3$ on both sides of (ii).

$$1^3 + 2^3 + 3^3 + \ldots + k^3 + (k+1)^3 = \frac{k^2(k+1)^2}{4} + (k+1)^3$$

$$= (k+1)^2\left[\frac{k^2 + 4(k+1)}{4}\right] = \frac{(k+1)^2(k^2+4k+4)}{4}$$

$$= \frac{(k+1)^2(k^2+2)^2}{4} = \frac{(k+1)^2(k+1+1)^2}{4} \qquad \ldots(iii)$$

$\therefore$ P(n) is true for $n = k + 1$ i.e., P(k + 1) is true

$\therefore$ By Principle of mathematical induction, P(n) is true for all natrueal numbers n.

3. $1 + \frac{1}{1+2} + \frac{1}{1+2+3} + \ldots + \frac{1}{1+2+3+\ldots+n} = \frac{2n}{n+1}$

Sol. Let the given statement be P (n).

$$\text{i.e. } P(n) : 1 + \frac{1}{1+2} + \frac{1}{1+2+3} + \ldots + \frac{1}{1+2+3+\ldots+n} = \frac{2n}{n+1}$$

$$\text{i.e. } P(n) : 1 + \frac{1}{1+2} + \frac{1}{1+2+3} + \ldots + \frac{2}{n(n+1)} = \frac{2n}{n+1} \qquad \ldots(i)$$

For $n = 1$. LHS = 1, RHS $\frac{2\times 1}{1+1} = \frac{2}{2} = 1$

$\because$ RHS = RHS $\therefore$ P(1) is true.

Let it is true for $n = k$ also.

$$P(k): 1 + \frac{1}{1+2} + \frac{1}{1+2+3} + \ldots + \frac{2}{k(k+1)} = \frac{2k}{k+1} \qquad \ldots(ii)$$

For $n = k + 1$,

$$P(k+1):\left(1+\frac{1}{1+2}+\frac{1}{1+2+3}+\ldots+\frac{2}{k(k+1)}\right)+\frac{2}{(k+1)\,(k+2)}=\frac{2(k+1)}{k+1+1}$$

adding $\frac{2}{(k+1)\,(k+2)}$ on both sides of (ii)

$$1+\frac{1}{1+2}+\ldots..+\frac{2}{(k+1)(k+2)}=\frac{2k}{k+1}+\frac{2}{(k+1)\,(k+2)}$$

$$=\frac{2\,[k^2+2k+1]}{(k+1)(k+2)}$$

$$=\frac{2(k+1)^2}{(k+1)\,(k+2)}$$

$$=\frac{2(k+1)}{k+2}=\frac{2(k+1)}{(k+1)+1}$$

Therefore, P(k + 1) is true when P(k) is true.

∴ By Principle of mathematical induction, P(n) is true $\forall\ n \in N$

4. 1. 2. 3 + 2. 3. 4 + … + $n(n+1)(n+2)$ $=\frac{n(n+1)(n+2)(n+3)}{4}$

Sol. Let P(n) : 1. 2. 3 + 2. 3. 4 + ...+ $n(n+1)(n+2)$

$$=\frac{n(n+1)(n+2)(n+3)}{4} \quad \ldots(i)$$

For n = 1,

LHS = 1. 2. 3 = 6

and RHS $=\frac{1(1+1)(1+2)(1+3)}{4}=\frac{1\times2\times3\times4}{4}=6$

∴ LHS = RHS i.e., P(1) is true

∴ Putting $n = k$ in Eq. (i) we have

P(k): 1. 2. 3 + 2. 3. 4 + … + $k(k+1)(k+2)$ $=\frac{k(k+1)(k+2)(k+3)}{4}$ …(ii)

Adding $(k+1)(k+2)(k+3)$ to both sides of Eq. (ii), we have

1. 2. 3 + 2. 3. 4 + 3. 4. 5 + … + $k(k+1)(k+2)+(k+1)(k+2)(k+3)$

$$=\frac{k(k+1)(k+2)\,(k+3)}{4}+(k+1)(k+2)(k+3)$$

$$= \frac{k(k+1)(k+2)(k+3)}{4} + (k+1)(k+2)(k+3) = (k+1)(k+2)(k+3)\left(\frac{k}{4}+1\right)$$

$$= \frac{(k+1)(k+2)(k+3)(k+4)}{4} = \frac{(k+1)(k+1+1)(k+1+2)(k+1+3)}{4} \quad \ldots\text{(iii)}$$

∴ P(n) is true for $n = k + 1$ i.e. P(k + 1) is true.

∴ By Principle of mathematical induction, P(n) is true for all natural numbers n.

5. $1.3 + 2.3^2 + 3.3^3 + \ldots + n.3^n = \frac{(2n-1)3^{n+1}+3}{4}$

Sol. Let P(n) : $1.3 + 2.3^2 + 3.3^3 + \ldots + n.3^n = \frac{(2n-1)3^{n+1}+3}{4}$

Putting $n = 1$,

LHS = 1.3 = 3,

RHS $= \frac{(2-1)3^2+3}{4} = \frac{12}{4} = 3 = \text{LHS}$

This shows P(n) is true for $n = 1$.

Let P(n) be true for $n = k$.

∴ P(k) : $1.3 + 2.3^2 + 3.3^3 + \ldots + k.3^k = \frac{(2k-1)3^{k+1}+3}{4}$ … (i)

Adding $(k+1)3^{k+1}$ to both sides of Eq. (i), we get

LHS = $1.3 + 2.3^2 + 3.3^3 + \ldots + k.\,3^k + (k+1)\,3^{k+1}$

$$\text{RHS} = \frac{(2k-1).3^{k+1}+3}{4} + (k+1).3^{k+1} = \frac{(2k-1)3^{k+1}+3+4(k+1)3^{k+1}}{4}$$

$$= \frac{3^{k+1}[2k-1+4(k+1)]+3}{4} = \frac{3^{k+1}(6k+3)+3}{4} = \frac{(2k+1)3^{k+2}+3}{4}$$

$$= \frac{[2(k+1)-1]3^{(k+1)+1}+3}{3}$$

This shows P(n) is true for $n = k + 1$ i.e. P(k + 1) is tue whenever P(k) is true

Hence, P(n) is true for all $n \in$N.

6. $1.2 + 2.3 + 3.4 + \ldots + n.\,(n+1) = \left[\frac{n(n+1)(n+2)}{3}\right]$

Sol. Let P(n) : $1.2 + 2.3 + 3.4 + \ldots + n.\,(n+1) = \left[\frac{n(n+1)(n+2)}{3}\right]$

for $n = 1$,
LHS = 1.2 = 2;

$$\text{RHS} = \left[\frac{1(n+1)(n+2)}{3}\right] = \frac{1.2.3}{3} = 1.2 = 2$$

$\therefore$ LHS = RHS
$\therefore$ P(1) is true.
We assume that P(n) is true for $n = k$.

i.e. $1.2 + 2.3 + 3.4 + \ldots + k(k+1) = \left[\frac{k(k+1)(k+2)}{3}\right]$ ...(i)

Adding $(k+1)(k+2)$ on both sides of Eq. (i)
LHS = $1.2 + 2.3 + 3.4 + \ldots + k(k+1) + (k+1)(k+2)$

$$\text{RHS} = \frac{k(k+1)(k+2)}{3} + (k+1)(k+2)$$

$$= \frac{k(k+1)(k+2) + 3(k+1)(k+2)}{3} = \frac{(k+1)(k+2)(k+3)}{3}$$

$\therefore$ $1.2 + 2.3 + 3.4 + \ldots + (k+1)(k+2) = \frac{(k+1)(k+1+1)(k+1+2)}{3}$

Thus, P($k + 1$) is true whenever P(k) is true.
Hence, by principle of mathematical induction
P(n) is true for all values of $n \in$ N

7. $1 . 3 + 3 . 5 + 5 . 7 + \ldots + (2n-1)(2n+1) = \frac{n(4n^2+6n-1)}{3}$

Sol. Let P(n) be the given statement

i.e., P(n) : $1 . 3 + 3 . 5 + 5 . 7 + \ldots + (2n-1)(2n+1) = \frac{n(4n^2+6n-1)}{3}$

Putting $n = 1$,
LHS = 1 . 3 = 3

and RHS $= \frac{1(4.1^2+6.1-1)}{3} = \frac{4+6-1}{3} = \frac{9}{3} = 3$

LHS = RHS $\therefore$ P(n) is true for $n = 1$.
Assume that P(n) is true for $n = k$ i.e P(k) is true.

i.e. P(k) : $1 . 3 + 3 . 5 + 5 . 7 + \ldots + (2k-1)(2k+1) = \frac{k(4k^2+6k-1)}{3}$ ...(ii)

$\therefore$ Adding $(2k+1)(2k+3)$ on both sides of Eq. (ii)

∴ LHS $=1 . 3 + 3 . 5 + 5 . 7 + ... + (2k - 1)(2k + 1) + (2k + 1)(2k + 3)$

RHS $= \dfrac{k(4k^2 + 6k - 1)}{3} + (2k + 1)(2k + 3)$

$= \dfrac{(4k^3 + 6k^2 - k) + 3(2k+1)(2k+3)}{3} = \dfrac{4k^3 + 18k^2 + 23k + 9}{3}$

$= \dfrac{(k+1)(4k^2 + 14k + 9)}{3} = \dfrac{(k+1)(4(k+1)^2 + 6(k+1) - 1)}{3}$

Thus, P(n) is true for $n = k + 1$

∴ P(k + 1) is true whenever P(k) is true

Hence, by principle of mathematical induction P(n) is true for all $n \in$N.

8. $1.2 + 2.2^2 + 3.2^3 + ... + n.2^n = (n - 1)\, 2^{n+1} + 2$

Sol. Let the given statement be P(n).

i.e. P(n): $1.2 + 2.2^2 + 3.2^3 + ... + n.2^n = (n - 1)\, 2^{n+1} + 2$

For $n = 1$,

P (1) : $(1 - 1)\, 2^{1+1} + 2 = 0 + 2 = 2 = 1.2$

which is true.

Let it is true for $n = k$,

i.e. $1.2 + 2.2^2 + 3.2^3 + ... + k\,.\,2^k = (k - 1)\, 2^{k+1} + 2$ ...(i)

For $n = k + 1$, LHS $= (1.2 + 2.2^2 + 3.2^3 + ... + k.2^k) + (k + 1)\, 2^{k+1}$

$= [(k - 1)\, 2^{k+1} + 2] + (k + 1)\, 2^{k+1}$ [Using Eq. (i)]

$= (k - 1)\, 2^{k+1} + 2 + (k + 1)\, 2^{k+1}$

$= 2^{k+1}\,(2k) + 2$

$= 2^{(k+1)+1}\,(k) + 2 = [(k + 1) - 1]\, 2^{(k+1)+1} + 2 =$ RHS

Therefore, P(k + 1) is true when P(k) is true.

9. $\dfrac{1}{2} + \dfrac{1}{4} + \dfrac{1}{8} + ... + \dfrac{1}{2^n} = 1 - \dfrac{1}{2^n}$

Sol. Let the given statement be P(n)

i.e. $P(n): \dfrac{1}{2} + \dfrac{1}{4} + \dfrac{1}{8} + ... + \dfrac{1}{2^n} = 1 - \dfrac{1}{2^n}$

For $n = 1$,

$P(1): 1 - \dfrac{1}{2^1} = 1 - \dfrac{1}{2} = \dfrac{1}{2}$

which is true.

Let it is true for $n = k$,

$$P(k): \frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots+\frac{1}{2^k}=1-\frac{1}{2^k} \quad \ldots(i)$$

For $n = k + 1$,

$$\text{LHS} = \left(\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots+\frac{1}{2^k}\right)+\frac{1}{2^{k+1}}=1-\frac{1}{2^k}+\frac{1}{2^{k+1}} \quad \text{[Using Eq. (i)]}$$

$$=1-\frac{1}{2^k}+\frac{1}{2^k.\,2}$$

Taking $\frac{1}{2^k}$ common in last two terms,

$$=1-\frac{1}{2^k}\left(1-\frac{1}{2}\right)=1-\frac{1}{2^k}\times\frac{1}{2}=1-\frac{1}{2^{k+1}} = \text{RHS}$$

Therefore, P(k + 1) is true when P (k) is true. Hence, from the principle of mathematical induction, the statement is true for all natural numbers n.

10. $\frac{1}{2.5}+\frac{1}{5.8}+\frac{1}{8.11}+\ldots+\frac{1}{(3n-1)(3n+2)}=\frac{n}{6n+4}$

Sol. Let the given statement be P(n), i.e.

$$P(n): \frac{1}{2.5}+\frac{1}{5.8}+\frac{1}{8.11}+\ldots+\frac{1}{(3n-1)(3n+2)}=\frac{n}{6n+4}$$

Putting $n = 1$,

$$\text{LHS} = \frac{1}{2.5}=\frac{1}{10}$$

and $\text{RHS}=\frac{1}{6+4}=\frac{1}{10}=\text{LHS}$

$\therefore$ P(n) is true for $n = 1$

$\therefore$ P(n) is true for $n = k$, i.e. P(k) is true

i.e. $$\frac{1}{2.5}+\frac{1}{5.8}+\frac{1}{8.11}+\ldots+\frac{1}{(3k-1)(3k+2)}=\frac{k}{6k+4} \quad \ldots(i)$$

$$k^{\text{th}}\text{ term} = \frac{1}{(3k-1)(3k+2)}$$

$$\therefore \quad (k+1)^{\text{th}}\text{ term}=\frac{1}{[3(k+1)-1][3(k+1)+1]}=\frac{1}{(3k+2)(3k+5)}$$

Adding this term on both sides of Eq. (i)

$$\text{LHS} = \frac{1}{2.5}+\frac{1}{5.8}+\frac{1}{8.11}+\ldots\ldots\ldots+\frac{1}{(3k-1)(3k+2)}+\frac{1}{(3k+2)(3k+5)}$$

$$\text{RHS} = \frac{k}{6k+4}+\frac{1}{(3k+2)(3k+5)}=\frac{k}{2(3k+2)}+\frac{1}{(3k+2)(3k+5)}$$

$$=\frac{k(3k+5)+2}{2(3k+2)(3k+5)}=\frac{3k^2+5k+2}{2(3k+2)(3k+5)}$$

$$=\frac{(3k+2)(k+1)}{2(3k+2)(3k+5)}=\frac{(k+1)}{2(3k+5)}=\frac{(k+1)}{(6k+10)}=\frac{(k+1)}{6(k+1)+4}$$

This shows that P(n) is true for $n = k + 1$

$\therefore$ P(k + 1) is true whenever P(k) is true.

i.e. by principle of mathematical induction, P(n) is true for all $n \in$ N.

11. $\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}+\ldots+\frac{1}{n(n+1)\,(n+2)}=\frac{n(n+3)}{4(n+1)\,(n+2)}$

Sol. Let the given statement be P(n).

i.e. $P(n):\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}+\ldots+\frac{1}{n(n+1)(n+2)}=\frac{n(n+3)}{4(n+1)(n+2)}$

For n = 1,

$$\frac{1(1+3)}{4(1+1)\,(1+2)}=\frac{4}{4\times 2\times 3}=\frac{1}{1.2.3}$$

which is true.

Let it is true for $n = k$,

$$\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}+\ldots+\frac{1}{k(k+1)\,(k+2)}=\frac{k(k+3)}{4(k+1)(k+2)}$$

Let us suppose P (k) is true.

For $n = k + 1$

$$\text{LHS} = \left(\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}+\ldots+\frac{1}{k(k+1)(k+2)}\right)+\frac{1}{(k+1)(k+1+1)\,(k+2+1)}$$

$$=\frac{k(k+3)}{4(k+1)\,(k+2)}+\frac{1}{(k+1)(k+2)(k+3)} \quad \text{[Using Eq. (i)]}$$

$$=\frac{k(k+3)^2+4}{4(k+1)(k+2)\,(k+3)}$$

$$= \frac{k^3 + 6k^2 + 9k + 4}{4(k+1)(k+2)(k+3)}$$

$$= \frac{(k+1)(k^2 + 5k + 4)}{4(k+1)(k+2)(k+3)} = \frac{k^2 + 5k + 4}{4(k+2)(k+3)}$$

$$= \frac{k^2 + 4k + 4 + 4}{4(k+2)(k+3)} = \frac{k(k+4) + (k+4)}{4(k+2)(k+3)}$$

$$= \frac{(k+1)(k+4)}{4(k+2)(k+3)} = \frac{(k+1)[(k+1)+3]}{4[(k+1)+1][(k+1)+2]} = \text{RHS}$$

$\therefore$ P(k + 1) is true whenever P(k) is true.

$\therefore$ By principal of mathematical induction, P(n) us true $\forall\ n \in$ N.

12. $a + ar + ar^2 + ... + ar^{n-1} = \dfrac{a(r^n - 1)}{r - 1}$

Sol. Let the given statement be P(n).

i.e. $P(n): a + ar + ar^2 + ... + ar^{n-1} = \dfrac{a(r^n - 1)}{r - 1}$

For n = 1,

i.e. $P(1) = \dfrac{a(r^1 - 1)}{r - 1} = a, \Rightarrow$ P(1) is true

Let it is true for $n = k$,

i.e. $a + ar + ar^2 + ... + ar^{k-1} = \dfrac{a(r^k - 1)}{r - 1}$ ...(i)

Let $n = k + 1$,

LHS = $(a + ar + ar^2 + ... + ar^{k-1}) + ar^{k+1-1}$

$= \dfrac{a(r^k - 1)}{r - 1} + ar^k$ [Using Eq. (i)]

$$= \frac{a(r^k - 1) + ar^k(r - 1)}{r - 1}$$

Taking a common in numerator part.

$$= \frac{a[r^k - 1 + r^k . r^1 - r^k]}{r - 1}$$

$$= \frac{a[r^k - 1 + r^{k+1} - r^k]}{r - 1}$$

$$= \frac{a[r^{k+1} - 1]}{r - 1} = \text{RHS}$$

Therefore, P $(k + 1)$ is true when P(k) is true.

Hence P(n) is true $\forall\ n \in \text{N}$

13. $\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)\ldots\left(1+\frac{2n+1}{n^2}\right) = (n+1)^2$

Sol. Let the given statement be P(n).

i.e. $P(n):\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)\ldots\left(1+\frac{2n+1}{n^2}\right) = (n+1)^2$

For $n = 1$,

$P(1) = (1+1)^2 = 2^2 = 4 = \left(1+\frac{3}{1}\right)$ which is true

Let $n = k$

$$\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)\ldots\left(1+\frac{2k+1}{k^2}\right) = (k+1)^2 \qquad \ldots\text{(i)}$$

Suppose P(k) is true.

For $n = k + 1$,

$$\text{LHS} = \left\{\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)\ldots\left(1+\frac{2k+1}{k^2}\right)\right\}\left(1+\frac{2(k+1)+2}{(k+1)^2}\right)$$

$$= (k+1)^2\left(1+\frac{2k+3}{(k+1)^2}\right) \qquad \text{[Using Eq. (i)]}$$

$$= (k+1)^2\left[\frac{(k+1)^2+2k+3}{(k+1)^2}\right] = k^2+2k+1+2k+3$$

i.e. $(k + 2)^2 = [(k + 1) + 1]^2 = \text{RHS}$

Therefore, P$(k + 1)$ is true whenever P(k) is true.

Hence, P(n) is true $\forall\ n \in \text{N}$

14. $\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\ldots\ldots\ldots\left(1+\frac{1}{n}\right) = (n+1)$

Sol. Let the given statement be P(n).

i.e $P(n):\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\ldots\ldots\ldots\left(1+\frac{1}{n}\right)=(n+1)$

For $n = 1$

$P(1):\ 1+1=1+\frac{1}{2}$, which is true,

Let it is true for $n = k$,

i.e., $\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\ldots\left(1+\frac{1}{k}\right)=k+1$ ...(i)

For $n = k + 1$

$$\text{LHS} = \left[\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\ldots\left(1+\frac{1}{k}\right)\right]\left(1+\frac{1}{k+1}\right)$$

$= (k+1)\left(1+\frac{1}{k+1}\right)$ [Using Eq. (i)]

$= (k+1)\left(\frac{k+1+1}{k+1}\right) = (k+1)+1$ = RHS

i.e. $(k+2)^2 = [(k+1)+1]^2$ = RHS

Therefore, P(k + 1) is true when P (k) is true, Hence, from the principle of mathematical induction, the statement is true for all natural numbers n.

15. $1^2+3^2+5^2+\ldots+(2n-1)^2=\frac{n(2n-1)(2n+1)}{3}$

Sol. Let the given statement be P(n).

$P(n): 1^2+3^2+5^2+\ldots+(2n+1)^2=\frac{n(2n-1)(2n+1)}{3}$

For $n = 1$,

$P(1):\frac{1(2-1)(2+1)}{3}=\frac{1\times1\times3}{3}=1=1^2$, which is true.

Let us suppose P (k) is true

$1^2+3^2+5^2+\ldots+(2k-1)^2=\frac{k(2k-1)(2k+1)}{3}$ ...(i)

For $n = k + 1$,

LHS $= [1^2 + 3^2 + 5^2 + \ldots + (2k-1)^2] + (2k-1+2)^2$

$= \frac{k(2k-1)(2k+1)}{3} + (2k+1)^2$ [Using eq. (i)]

$$= \frac{k(2k-1)\,(2k+1)+3(2k+1)^2}{3}$$

$$= \frac{(2k+1)\,[k\,(2k-1)+3(2k+1)]}{3}$$

$$= \frac{(2k+1)(2k^2+5k+3)}{3}$$

$$= \frac{(k+1)(2k+2-1)(2k+2+1)}{3}$$

$$= \frac{(k+1)\,[2(k+1)-1]\,[2(k+1)+1]}{3} = \text{RHS}$$

Therefore, P $(k + 1)$ is true when P (k) is true. Hence, from the principle of mathematical induction, the statement is true for all natural numbers n.

16. $\frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+\ldots+\frac{1}{(3n-2)(3n+1)}=\frac{n}{(3n+1)}$

Sol. Let the given statement be P(n).

$$P(n): \frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+\ldots+\frac{1}{(3n-2)(3n+1)}=\frac{n}{(3n+1)}$$

for $n = 1$

$$\text{LHS} = \frac{1}{1.4} = \frac{1}{4}$$

$$\text{and RHS} = \frac{1}{3.1+1} = \frac{1}{4}$$

$\therefore$ P(n) is true for $n = 1$.

Assume that P(n) is true for $n = k$, i.e. P(k) is true.

$$\text{i.e., } \frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+\ldots+\frac{1}{(3k-2)\,(3k+1)}=\frac{k}{3k+1} \qquad \ldots\text{(i)}$$

For $n = k + 1$,

$$\text{LHS} = \left[\frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+\ldots+\frac{1}{(3k-2)\,(3k-1)}\right]+\frac{1}{(3k-2+3)\,(3k+1+3)}$$

$$= \frac{k}{3k+1}+\frac{1}{(3k+1)\,(3k+4)} \qquad \text{[Using Eq. (i)]}$$

$$= \frac{k\,(3k+4)+1}{(3k+1)(3k+4)} = \frac{3k^2+4k+1}{(3k+1)\,(3k+4)}$$

$$= \frac{(k+1)(3k+1)}{(3k+1)(3k+4)} = \frac{k+1}{3k+3+1} = \frac{k+1}{3(k+1)+1} = \text{RHS}$$

Therefore, P $(k+1)$ is true when P (k) is true. Hence, from the principle of mathematical induction, the statement is true for all natural numbers n.

17. $\frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}+...+\frac{1}{(2n+1)(2n+3)} = \frac{n}{3(2n+3)}$

Sol. Let the statement be denoted by P(n).

i.e. $$P(n): \frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}+...+\frac{1}{(2n+1)(2n+3)} = \frac{n}{3(2n+3)}$$

For $n = 1$,

$$P(1): \frac{1}{3(2\times1+3)} = \frac{1}{3\times5} = \frac{1}{3.5}$$ which is true.

Let it is true for $n = k$,

i.e. $$\frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}+...+\frac{1}{(2k+1)(2k+3)} = \frac{k}{3(2k+3)} \quad ...(i)$$

For $n = k+1$,

$$\text{LHS} = \left[\frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}+...+\frac{1}{(2k+1)(2k+3)}\right] + \frac{1}{(2k+1+2)(2k+3+2)}$$

$$= \frac{k}{3(2k+3)} + \frac{1}{(2k+3)(2k+5)}$$

$$= \frac{k(2k+5)+3}{3(2k+3)(2k+5)} = \frac{2k^2+5k+3}{3(2k+3)(2k+5)}$$

$$= \frac{(k+1)(2k+3)}{3(2k+3)(2k+5)} = \frac{k+1}{3(2k+2+3)} = \frac{k+1}{3(2(k+1)+3)} = \text{RHS}$$

Therefore, $P(k+1)$ is true when P (k) is true. Hence, from the principle of mathematical induction, the statement is true for all natural numbers n.

18. $1+2+3+...+n < \frac{1}{8}(2n+1)^2$

Sol. Let the given statement be P(n)

i.e. $$P(n): 1+2+3+...+n < \frac{1}{8}(2n+1)^2$$

For $n = 1$,

$$1 < \frac{1}{8}(2.1+1)^2 \Rightarrow 1 < \frac{1}{8} \times 3^2 \Rightarrow 1 < \frac{9}{8}$$

which is true

Let it is true for $n = k$,

$$1+2+3+...+k < \frac{1}{8}(2k+1)^2 \quad ...(i)$$

For $n = k + 1$,

$$(1+2+3+...+k)+(k+1) < \frac{1}{8}(2k+1)^2 + (k+1) \quad \text{[Using Eq. (i)]}$$

$$= \frac{(2k+1)^2}{8} + \frac{k+1}{1} = \frac{(2k+1)^2 + 8k + 8}{8}$$

$$= \frac{(2k+3)^2}{8} = \frac{(2k+2+1)^2}{8} = \frac{[2(k+1)+1]^2}{8}$$

$$\Rightarrow 1+2+3+...+k+(k+1) < \frac{[2(k+1)+1]^2}{8}$$

Therefore, P $(k + 1)$ is true when P(k) is true.

19. $n(n + 1)(n + 5)$ is a multiple of 3.

Sol. Let the statement be denoted by P(n)

i.e. P(n) : $n(n + 1)(n + 5)$ is a multiple of 3.

For $n = 1$,

$n(n+ 1)(n + 5) = 1.\ 2.\ 6 = 12$ which is a multiple of 3.

P(n) is true for $n = 1$

Suppose P(k) is true for $n = k$

i.e. $k(k + 1)(k + 5)$ is a multiple of 3.

or, $k(k + 1)(k + 5) = 3m,\ m \in$ N

or, $k^3 + 6k^2 + 5k = 3m$, where $m \in$ N … (i)

Replacing k by $k + 1$, we get

$(k + 1)(k + 2)(k + 6) = k(k^2 + 8k + 12) + (k^2 + 8k + 12)$

$= k^3 + 9k^2 + 20k + 12 = (k^3 + 6k^2 + 5k) + (3k^2 + 15k + 12)$

$= 3m + 3k^2 + 15k + 12$ [from (i)]

$=3(m + k^2 + 5k + 4) = 3$ where $p = (m + k^2 + 5k + 4)$, $k \in$ N.

i.e. $(k + 1)(k + 2)(k + 6)$ is a multiple of 3

i.e. P$(k + 1)$ is multiple of 3, if P(k) is a multiple of 3

i.e. P$(k + 1)$ is true whenever P(k) is true.

Hence P(n) is true for all $n \in N$

20. $10^{2n-1} + 1$ is divisible by 11

Sol. Let P(n) : $10^{2n-1} + 1$ is divisible by 11 for every natural number n.

for $n = 1$,

$P(1) = 10^{2-1} + 1 = (10 + 1) = 11$ which is divisible by 11

$\therefore$ P(1) is true.

Let it is true for $n = k$,Put $n = k$ in Eq. (i), we have $10^{2k-1} + 1$ is divisible by 11.

$\therefore \quad 10^{2k-1} + 1 = 11\,m, \;\; m \in N$

$\Rightarrow \quad 10^{2k-1} = 11m - 1$ ...(ii)

Putting $n = k + 1$ in $10^{2n-1} + 1$, it becomes

$10^{2(k+1)-1} + 1 = (10^{2k-1}.\ 10^2) + 1$

$= 100.\ 10^{2k-1} + 1 = 100(11m - 1) + 1$ (using (ii))

$= 100 \times 11m - 100 + 1 = 100 \times 11m - 99$

$= 11(100m - 9)$, which is divisible by 11 [$\because$ 11 is a factor of RHS]

$\therefore \quad 10^{2(k+1)-1} + 1$ is divisible by 11

$\therefore$ P(n) is true for $n = k + 1$ i.e. P($k + 1$) is true ...(iii)

$\therefore$ By principle of mathematical induction, P(n) is true for all natural numbers n.

21. $x^{2n} - y^{2n}$ is divisible by $x + y$.

Sol. Let the statement be P(n), i.e. P(n) : $x^{2n} - y^{2n}$ is divisible by $x + y$. ... (i)

Putting $n = 1$, $x^{2n} - y^{2n} = x^2 - y^2 = (x + y)(x - y)$

which is divisible by $x + y$ $\Rightarrow$ P(n) is true for $n = 1$

Let P(k) be true i.e. $x^{2k} - y^{2k}$ is divisible by $x + y$.

or, $x^{2k} - y^{2k} = m(x + y)$ where $m \in N$ or $x^{2k} = m(x + y) + y^{2k}$ ... (ii)

Replace k by $k + 1$ in $x^{2k} - y^{2k}$, we get

$x^{2(k+1)} - y^{2(k+1)} = x^{2k+2} - y^{2k+2} = x^2 . x^{2k} - y^{2k+2}$

Putting the value of x^{2k} from Eq. (ii),

$= x^2 [m(x + y) + y^{2k}] - y^{2k+2} = m(x + y)\,x^2 + x^2y^{2k} - y^{2k+2}$

$= m(x + y)\,x^2 + y^{2k}(x^2 - y^2) = m(x + y)\,x^2 + (x + y)(x - y)\,y^{2k}$

$=(x + y)\,[mx^2 + (x - y)y^{2k}] = p(x + y)$, is divisible by $x + y$.

(where, $p = mx^2 + (x - y)y^{2k}$)

Therefore, P($k + 1$) is true whenever P(k) is true.

Hence P(n) is true $\forall\ n \in N$.

22. $3^{2n+2} - 8n - 9$ is divisible by 8.

Sol. Let the statement be P(n) i.e

P(n) : $3^{2n+2} - 8n - 9$ is divisible by 8.

For $n = 1$,

i.e., P(1) : $3^{2\times1+2} - 8 \times 1 - 9 = 3^4 - 8 - 9 = 81 - 17$

$= 64 = 8 \times 8$

which is divisible by 8.

Let it is true for $n = k$,

$P(k) = 3^{2k+2} - 8k - 9 = 8\lambda$ $\lambda \in N$

$3^{2k+2} = 8\lambda + 8k + 9$ ...(i)

For P (k + 1) :

$3^{2(k+1)+2} - 8(k+1) - 9$

$= 3^{2k+2+2} - 8k - 8 - 9 = 3^{2k+2}\, 3^2 - 8k - 17$ [Using Eq. (i)]

$=(8\lambda + 8k + 9)\, 3^2 - 8k - 17$

$= 72\lambda + 64k + 64 = 8(9\lambda + 8k + 8)$

which is divisible by 8.

(where, $p = mx^2 + (x-y)y^{2k}$)

Therefore, P (k + 1) is true when P (k) is true.

$\therefore$ P(n) is true $\forall\ n \in N$

23. $41^n - 14^n$ is a multiple of 27.

Sol. Let the given statement be P(n).

i.e., P(n): $41^n - 14^n$ is a multiple of 27.

For $n = 1$,

$P(1) = 41^1 - 14^1 = 27 = 1 \times 27$

which is a multiple of 27 that is true.

Let it is true for n = k,

i.e., $41^k - 14^k$ is a multiple of 27

or $41^k - 14^k = 27\lambda$ ($\lambda \in N$)

$41^k = 27\lambda + 14^k$ ...(i)

For $n = k + 1$, $41^{k+1} - 14^{k+1}$

$= 41^k\, 41 - 14^k\, 14$

$= (27\lambda + 14^k)\, 41 - 14^k\, 14$ [Using Eq. (i)]

$= 27\lambda + 41 + 14^k \times 27 = 27(41\lambda + 14^k)$

which is a multiple of 27.

Therefore, P $(k + 1)$ is true P (k) is true.

24. $(2n + 7) < (n + 3)^2$.

Sol. Let the given statement be P(n), i.e.

P(n) : $(2n + 7) < (n + 3)^2$

For n = 1,

P(1) : $2 \times 1 + 7 < (1 + 3)^2 = 9 < 16$

which is true

For $n = k$, $2k + 7 < (k + 3)^2$ … (i)

Let it is true for $n = k$

Let $n = k + 1$

$$2(k + 1) + 7 < (k + 1 + 3)^2$$

$$2k + 9 < (k + 4)^2$$

$$2k + 9 < k^2 + 8k + 16$$

Rewrite Eq. (i)

$$2k + 7 < (k + 3)^2$$

Add 2 on both sides –

$$2k + 9 < k^2 + 6k + 11$$

Hence, $2k + 9 < k^2 + 6k + 11 < k^2 + 8k + 16$

$$2k + 9 < k^2 + 8k + 16$$

Therefore, P $(k + 1)$ is true when P (k) is true.

Hence, P(n) is true $\forall\ n \in$ N.