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Sample Paper

(2018-19)

Date : ___________ **Duration : 3 Hrs.** **Max. Marks : 100**	**Mathematics** (Set-1)	**Class** **XII**

Instructions:

- All questions are compulsory.
- Do not write anything in the question paper.
- Questions No. 1 to 4 carry 1 mark each.
- Questions No. 5 to 12 carry 2 marks each.
- Questions No. 13 to 23 carry 4 marks each.
- Questions No. 24 to 29 carry 6 marks each.
- Use of calculator is not permitted.

Section : A

1. Write the value of $\tan\left(2\tan^{-1}\frac{1}{5}\right)$.

2. If $A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & x \\ -2 & 2 & -1 \end{pmatrix}$ is a matrix satisfying AA′ = 9I, find x.

OR

Find values of k if area of triangle is 4 sq. units with vertices $(k, 0)$, $(4, 0)$, $(0, 2)$

3. Find the second order derivative of the function $f(x) = x^3 \log x$ with respect to x.

4. If $|\vec{a}| = 8, |\vec{b}| = 3$ and $|\vec{a} \times \vec{b}| = 12$, then find the value of $|\vec{a}.\vec{b}|$.

Section : B

5. Prove that $\tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{8} = \frac{\pi}{4}$

OR

Find the value of $\tan^{-1}\left(\frac{\cos x - \sin x}{\cos x + \sin x}\right)$, $0 < x < \pi$.

6. Find $\int \tan^8 x \sec^4 x dx$

7. A balloon, which always remains spherical, has a variable diameter $\frac{3}{2}(2x + 1)$. Find the rate of change of its volume with respect to x.

8. If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, show that $A^2 - 5A + 7I = 0$.

9. Differentiate w.r.t. : $\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$.

OR

Differentiate w.r.t. x the function $(\sin x - \cos x)^{\sin x - \cos x}$, $\frac{\pi}{4} < x < \frac{3\pi}{4}$

10. Solve the differential equation $\cos\left(\frac{dy}{dx}\right) = a, (a \in \mathbb{R})$.

11. If $\vec{a}+\vec{b}+\vec{c}=\vec{0}$ and $|\vec{a}|=5, |\vec{b}|=6$ and $|\vec{c}|=9,$ then find the angle between $\vec{a}$ and $\vec{b}$.

12. A die is tossed thrice. Find the probability of getting an odd number at least once.

OR

Assume that each born child is equally likely to be a boy or a girl. If a family has two children, what is the conditional probability that both are girls given that (i) the youngest is a girl, (ii) at least one is a girl?

Section : C

13. If a, b and c are real numbers, and $\Delta = \begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = 0,$

Show that either $a + b + c = 0$ or $a = b = c$.

OR

Prove that $\begin{vmatrix} \sin\alpha & \cos\alpha & \cos(\alpha+\delta) \\ \sin\beta & \cos\beta & \cos(\beta+\delta) \\ \sin\gamma & \cos\gamma & \cos(\gamma+\delta) \end{vmatrix} = 0$

14. If $y = e^{a\cos^{-1}x}$, $-1 \le x \le 1$, show that $(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - a^2y = 0$.

OR

If $x = a \sin pt$, $y = b \cos pt$, then find $\frac{d^2y}{dx^2}$ at $t = 0$.

15. Find : $\int \frac{5x+3}{\sqrt{x^2+4x+10}}dx$

OR

Find the following indefinite integral

$$\int \frac{x-1}{(x+1)\sqrt{x^3+x^2+x}}dx.$$

16. Show that the height of the cylinder of maximum volume that can be inscribed in a sphere of radius R is $\frac{2R}{\sqrt{3}}$. Also find the maximum volume.

17. Find all the points of discontinuity of the function f defined by $f(x) = |x| - |x + 1|$.

18. Solve the differential equation $\left(\frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}\right)\frac{dx}{dy} = 1 \ (x \neq 0)$.

19. Find the intervals in which the function f given by $f(x) = 2x^3 - 3x^2 - 36x + 7$ is

(a) strictly increasing (b) strictly decreasing

20. Find the shortest distance between the lines $\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1}$ and $\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1}$.

21. If $\vec{a} = \hat{i}+\hat{j}+\hat{k}$ and $\vec{b} = \hat{j}-\hat{k},$ then find a vector $\vec{c}$ such that $\vec{a}\times\vec{c} = \vec{b}$ and $\vec{a}.\vec{c} = 3$.

22. Given three identical boxes I, II and III, each containing two coins. In box I, both coins are gold coins, in box II, both are silver coins and in the box III, there is one gold and one silver coin. A person chooses a box at random and takes out a coin. If the coin is of gold, what is the probability that the other coin in the box is also of gold?.

23. Find the real solutions of the equation $\tan^{-1}\left(\frac{1-x}{1+x}\right)=\frac{1}{2}\tan^{-1}x, (x>0)$.

Section : D

24. Consider $f: \mathbf{R}_+ \to [-5, \infty)$ given by $f(x) = 9x^2 + 6x - 5$. Show that f is invertible with $f^{-1}(y)=\left(\frac{\left(\sqrt{y+6}\right)-1}{3}\right)$.

OR

Show that each of the relation R in the set A = $\{x \in \mathbf{Z} : 0 \leq x \leq 12\}$, given by

(i) R = $\{(a, b) : |a - b|$ is a multiple of 4$\}$

is an equivalence relation. Find the set of all elements related to 1.

25. Given $A=\begin{bmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{bmatrix}, B=\begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix}$, find BA and use this to solve the system of equations $y + 2z = 7$, $x - y = 3$, $2x + 3y + 4z = 17$.

OR

If possible, using elementary row transformation, find the inverse of the following matrix

$$A=\begin{bmatrix} 2 & -1 & 3 \\ -5 & 3 & 1 \\ -3 & 2 & 3 \end{bmatrix}$$

26. Evaluate $\int_0^{\pi} \frac{x \tan x}{\sec x + \tan x}\, dx$

OR

Evaluate the integral as a limit of sums : $\int_1^3 (e^{2-3x} + x^2 + 1)dx$

27. Find the area of the smaller region bounded by the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ and the line $\frac{x}{3}+\frac{y}{2}=1$.

28. Find the equation of the plane which contains the line of intersection of the planes $\vec{r}\cdot(\hat{i}+2\hat{j}+3\hat{k})-4=0$, $\vec{r}\cdot(2\hat{i}+\hat{j}-\hat{k})+5=0$ and which is perpendicular to the plane $\vec{r}\cdot(5\hat{i}+3\hat{j}-6\hat{k})+8=0$.

29. An aeroplane can carry a maximum of 200 passengers. A profit of ₹1,000 is made on each executive class ticket and a profit of ₹600 is made on each economy class ticket. The airline reserves at least 20 seats for executive class. However, at least 4 times as many passengers prefer to travel by economy class than by the executive class. Determine how many tickets of each type must be sold in order to maximise the profit for the airline. What is the maximum profit?

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Hints/Solutions to Sample Paper

(2018-19)

Date : ________ Duration : 3 Hrs. Max. Marks : 100	**Mathematics** (Set-1)	Class XII

Section-A

1. $\tan\left[\tan^{-1}\left(\dfrac{\frac{2}{5}}{1-\frac{1}{25}}\right)\right]$

$\tan\left[\tan^{-1}\left(\dfrac{10}{24}\right)\right] = \dfrac{5}{12}$

2. $A' = \begin{pmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ 2 & x & -1 \end{pmatrix}$ and getting $x = -2$

OR

$(k, 0), (4, 0), (0, 2)$

$\Delta = 4$ sq. units.

$\Rightarrow \dfrac{1}{2}\begin{vmatrix} k & 0 & 1 \\ 4 & 0 & 1 \\ 0 & 2 & 1 \end{vmatrix} = \pm 4$

$\Rightarrow \dfrac{1}{2}[-2k+8] = \pm 4$

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$-2k + 8 = 8$ $-2k + 8 = -8$

$k = 0$ $k = 8$

$\therefore k = 0, 8$

3. Let $y = x^3 \log x$

Then, $\dfrac{dy}{dx} = x^3 \cdot \dfrac{1}{x} + \log x \times 3x^2 = x^2 + 3x^2 \log x$

and $\dfrac{d^2y}{dx^2} = 2x + 3x^2 \cdot \dfrac{1}{x} + \log x \,.\, 6x = 2x + 3x + 6x \log x = 5x + 6x \log x$

$\therefore \dfrac{d^2y}{dx^2} = x(5 + 6 \log x)$

4. We have $|\vec{a}\times\vec{b}| = |\vec{a}||\vec{b}||\sin\theta|$ then $\theta = \dfrac{\pi}{6}, \dfrac{5\pi}{6}$

Therefore $|\vec{a}.\vec{b}| = |\vec{a}||\vec{b}||\cos\theta| = 8 \times 3 \times \dfrac{\sqrt{3}}{2} = 12\sqrt{3}.$

Section : B

5. L.H.S. = $\tan^{-1}\dfrac{1}{5} + \tan^{-1}\dfrac{1}{7} + \tan^{-1}\dfrac{1}{3} + \tan^{-1}\dfrac{1}{8} = \tan^{-1}\left(\dfrac{\frac{1}{5}+\frac{1}{7}}{1-\frac{1}{5}\times\frac{1}{7}}\right) + \tan^{-1}\left(\dfrac{\frac{1}{3}+\frac{1}{8}}{1-\frac{1}{3}\times\frac{1}{8}}\right)$

$$= \tan^{-1}\left(\frac{\frac{12}{35}}{\frac{34}{35}}\right) + \tan^{-1}\left(\frac{\frac{11}{24}}{\frac{23}{24}}\right) = \tan^{-1}\frac{12}{34} + \tan^{-1}\frac{11}{23}$$

$$= \tan^{-1}\frac{6}{17} + \tan^{-1}\frac{11}{23} = \tan^{-1}\left(\frac{\frac{6}{17}+\frac{11}{23}}{1-\frac{6}{17}\times\frac{11}{23}}\right) = \tan^{-1}\left(\frac{\frac{138+187}{391}}{\frac{391-66}{391}}\right)$$

$$= \tan^{-1}\left(\frac{325}{325}\right) = \tan^{-1}(1) = \frac{\pi}{4}.$$

= R.H.S

OR

The function is $\tan^{-1}\left(\frac{\cos x - \sin x}{\cos x + \sin x}\right)$

Dividing numerator and denominator by cos x,

$$= \tan^{-1}\left(\frac{\frac{\cos x}{\cos x} - \frac{\sin x}{\cos x}}{\frac{\cos x}{\cos x} + \frac{\sin x}{\cos x}}\right) = \tan^{-1}\left(\frac{1-\tan x}{1+\tan x}\right)$$

[Since, $\tan\left(\frac{\pi}{4} - \text{A}\right) = \frac{\tan\frac{\pi}{4} - \tan \text{A}}{1 + \tan\frac{\pi}{4}\tan \text{A}} = \frac{1-\tan \text{A}}{1+\tan \text{A}}$]

$$= \tan^{-1}\left[\tan\left(\frac{\pi}{4} - x\right)\right] = \frac{\pi}{4} - x.$$

6. $\text{I} = \int \tan^8 x \sec^4 x dx$

$$= \int \tan^8 x\left(\sec^2 x\right)\sec^2 x dx$$

$$= \int \tan^8 x(\tan^2 x + 1)\sec^2 x dx$$

$$= \int \tan^{10} x \sec^2 x dx + \int \tan^8 x \sec^2 x dx$$

$$= \frac{\tan^{11} x}{11} + \frac{\tan^9 x}{9} + C$$

7. Diameter of the sphere $= \frac{3}{2}(2x + 1)$

$\therefore$ Radius of the sphere $= \frac{3}{4}(2x+1)$

Volume of the sphere,

$$V = \frac{4}{3}\pi \cdot \frac{27}{64}(2x+1)^3 = \frac{9\pi}{16}(2x+1)^3$$

$\therefore$ Rate of change of volume, $\frac{dV}{dx} = \frac{9\pi}{16} \cdot 3\,(2x+1)^2 \cdot 2$

$$\frac{dV}{dx} = \frac{27\pi}{8}(2x+1)^2$$

8. $\text{A}^2 = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 9-1 & 3+2 \\ -3-2 & -1+4 \end{bmatrix} = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix}$

L.H.S. $= A^2 - 5A + 7I = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} - 5\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} + \begin{bmatrix} -15 & -5 \\ 5 & 10 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 8-15+7 & 5-5+0 \\ -5+5+0 & 3-10+7 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

= R.H.S.

Hence $A^2 - 5A + 7I = 0$

9. Let $y = \sqrt{\dfrac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$

Taking *log* on both sides, we get

$$\log y = \log \sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}} = \frac{1}{2}\log \frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}$$

$$\log y = \frac{1}{2}[\log(x-1) + \log(x-2) - \log(x-3) - \log(x-4) - \log(x-5)]$$

Differentiating w.r.t. x, $\dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{1}{2}\left[\dfrac{1}{x-1} + \dfrac{1}{x-2} - \dfrac{1}{x-3} - \dfrac{1}{x-4} - \dfrac{1}{x-5}\right]$

$$\therefore \frac{dy}{dx} = \frac{y}{2}\left[\frac{1}{x-1} + \frac{1}{x-2} - \frac{1}{x-3} - \frac{1}{x-4} - \frac{1}{x-5}\right]$$

$$\therefore \frac{dy}{dx} = \frac{1}{2}\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}\left[\frac{1}{x-1} + \frac{1}{x-2} - \frac{1}{x-3} - \frac{1}{x-4} - \frac{1}{x-5}\right]$$

OR

Let $y = (\sin x - \cos x)^{\sin x - \cos x}$

Taking *log* on both sides, we get

$\log y = (\sin x - \cos x)\log(\sin x - \cos x)$

Differentiating w.r.t. x, we get

$$\frac{1}{y}\frac{dy}{dx} = (\cos x + \sin x)\log(\sin x - \cos x) + (\sin x - \cos x)\frac{d}{dx}\log(\sin x - \cos x)$$

$$= (\cos x + \sin x)\log(\sin x - \cos x) + (\sin x - \cos x)\frac{1}{(\sin x - \cos x)}(\cos x + \sin x)$$

$$= (\cos x + \sin x)\log(\sin x - \cos x) + (\cos x + \sin x)$$

$$= (\cos x + \sin x)[\log(\sin x - \cos x) + 1]$$

$$\therefore \frac{dy}{dx} = (\sin x - \cos x)^{(\sin x - \cos x)}(\sin x + \cos x)[1 + \log(\sin x - \cos x)],\ \sin x > \cos x$$

10. $\dfrac{dy}{dx} = \cos^{-1} a \Rightarrow \int dy = \cos^{-1} a \cdot \int dx$

$y = x\cos^{-1} a + c$

11. $\vec{a} + \vec{b} + \vec{c} = 0$

$\Rightarrow |\vec{a} + \vec{b}|^2 = |\vec{c}|^2$

$\Rightarrow |\vec{a}|^2 + |\vec{b}|^2 + 2\vec{a}\cdot\vec{b} = |\vec{c}|^2$

$\Rightarrow \vec{a}\cdot\vec{b} = \dfrac{|\vec{c}|^2 - |\vec{a}|^2 - |\vec{b}|^2}{2}$

Also, $\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \dfrac{|\vec{c}|^2 - |\vec{a}|^2 - |\vec{b}|^2}{2|a||b|}$

$= \dfrac{9^2 - 5^2 - 6^2}{2(5)(6)}$

$\cos\theta = \dfrac{81 - 25 - 36}{60} = \dfrac{1}{3}$

$\theta = \cos^{-1}\left(\dfrac{1}{3}\right)$

12. Probability of getting an odd number in a single throw of a die = $\dfrac{3}{6} = \dfrac{1}{2}$

Similarly, probability of getting an even number = $\dfrac{3}{6} = \dfrac{1}{2}$

Probability of getting an even number three times = $\dfrac{1}{2}\times\dfrac{1}{2}\times\dfrac{1}{2} = \dfrac{1}{8}$

Therefore, probability of getting an odd number at least once

= 1 – Probability of getting an odd number in none of the throws

= 1 – Probability of getting an even number thrice

$= 1 - \dfrac{1}{8} = \dfrac{7}{8}$

OR

Let b and g represent the boy and the girl child respectively. If a family has two children, the sample space will be

S = {(b, b), (b, g), (g, b), (g, g)}

Let A be the event that both children are girls.

$\therefore$ A = {(g, g)}

(i) Let B be the even that the youngest child is a girl

$\therefore$ B = {(b, g), (g , g)}

$\Rightarrow A \cap B$ = {(g, g)}

$\therefore P(B) = \frac{2}{4} = \frac{1}{2}$

$P(A \cap B) = \frac{1}{4}$

The conditional probability that both are girls, given that the youngest child is a girl, is given by P(A|B).

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}$$

Therefore, the required probability is $\frac{1}{2}$

(ii) Let be the even that at least one child is a girl.

$\therefore$ C = {(b, g), (g, b), (g, g)}

$\Rightarrow A \cap C$ = {(g, g)}

$\Rightarrow P(C) = \frac{3}{4}$

$P(A \cap C) = \frac{1}{4}$

The conditional probability that both are girls, given that at least one child is a girl, is given by P(A|C).

Therefore, $P(A|C) = \frac{P(A \cap C)}{P(C)} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}$

Section-C

13. $\Delta = \begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix}$

Operating $C_1 \to C_1 + C_2 + C_3$

$$\Delta = \begin{vmatrix} 2(a+b+c) & c+a & a+b \\ 2(a+b+c) & a+b & b+c \\ 2(a+b+c) & b+c & c+a \end{vmatrix} \Rightarrow \Delta = 2\begin{vmatrix} a+b+c & c+a & a+b \\ a+b+c & a+b & b+c \\ a+b+c & b+c & c+a \end{vmatrix}$$

Operating $C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_1$

$$\Delta = 2\begin{vmatrix} a+b+c & -b & -c \\ a+b+c & -c & -a \\ a+b+c & -a & -b \end{vmatrix}$$

Taking out $(a + b + c)$ common from C_1, (– 1) from C_2 and C_3,

$$\Delta = 2(a+b+c)\begin{vmatrix} 1 & b & c \\ 1 & c & a \\ 1 & a & b \end{vmatrix}$$

Expanding with elements of 1st column

$\Delta = 2(a + b + c)[(cb - a^2) - (b^2 - ac) + (ab - c^2)]$

$= 2(a + b + c)[-(a^2 - bc) - (b^2 - ca) - (c^2 - ab)$

$= -(a + b + c)[2a^2 + 2b^2 + 2c^2 - 2bc - 2ca - 2ab]$

$= -(a + b + c)[(a - b)^2 + (b - c)^2 + (c - a)^2]$

Now, $\Delta = 0$, when either $(a + b + c) = 0$

or $(a - b)^2 + (b - c)^2 + (c - a)^2 = 0 \Rightarrow a = b = c$

Hence $\Delta = 0$ when $a + b + c = 0$ or $a = b = c$.

OR

Let $\Delta = \begin{vmatrix} \sin\alpha & \cos\alpha & \cos(\alpha+\delta) \\ \sin\beta & \cos\beta & \cos(\beta+\delta) \\ \sin\gamma & \cos\gamma & \cos(\gamma+\delta) \end{vmatrix}$

Using $\cos(A + B) = \cos A \cos B - \sin A \sin B$,

$$\Delta = \begin{vmatrix} \sin\alpha & \cos\alpha & \cos\alpha\cos\delta - \sin\alpha\sin\delta \\ \sin\beta & \cos\beta & \cos\beta\cos\delta - \sin\beta\sin\delta \\ \sin\gamma & \cos\gamma & \cos\gamma\cos\delta - \sin\gamma\sin\delta \end{vmatrix}$$

Operating $C_3 \to C_3 + \sin\delta \times C_1 - \cos\delta \times C_2$

$$= \begin{vmatrix} \sin\alpha & \cos\alpha & 0 \\ \sin\beta & \cos\beta & 0 \\ \sin\gamma & \cos\gamma & 0 \end{vmatrix} = 0$$

14. We have $y = e^{a\cos^{-1}x}$

Differentiate w.r.t. x, $\dfrac{dy}{dx} = y_1 = e^{a\cos^{-1}x}\dfrac{-a}{\sqrt{1-x^2}}$

$\Rightarrow \sqrt{1-x^2}\, y_1 = -ae^{a\cos^{-1}x}$

Again differentiate w.r.t. x, we get

$$\sqrt{1-x^2}\, y_2 + y_1 \cdot \frac{1}{2}\frac{1}{\sqrt{1-x^2}}(-2x) = \frac{a^2e^{a\cos^{-1}x}}{\sqrt{1-x^2}}$$

$\Rightarrow (1 - x^2)\, y_2 - xy_1 = a^2y \quad \Rightarrow (1 - x^2)\, y_2 - xy_1 - a^2y = 0.$

OR

$$\frac{dx}{dt} = ap\cos pt, \frac{dy}{dt} = -bp\sin pt, \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = -\frac{b}{a}\tan pt$$

$$\frac{d^2y}{dx^2} = -\frac{b}{a}p\sec^2 pt \times \frac{dt}{dx}$$

$$\frac{d^2y}{dx^2} = -\frac{b}{a^2\cos^3 pt}$$

$$\Rightarrow \left(\frac{d^2y}{dx^2}\right)_{t=0} = -\frac{b}{a^2}$$

15. Let $5x + 3 = A\dfrac{d}{dx}(x^2 + 4x + 10) + B$

$5x + 3 = A(2x + 4) + B.$

Comparing coefficient of x and constant on both side

We get, $A = \dfrac{5}{2}, B = -7.$

$$\int\frac{5x+3}{\sqrt{x^2+4x+10}}dx = \frac{5}{2}\int\frac{2x+4}{\sqrt{x^2+4x+10}} + (-7)\int\frac{dx}{\sqrt{x^2+4x+10}}$$

$$= \frac{5}{2}I_1 - 7I_2 \qquad \text{...(1)}$$

Now,

$$I_1 = \int\frac{2x+4}{\sqrt{x^2+4x+10}}dx$$

Let $x^2 + 4x + 10 = t$

$(2x + 4)dx = dt$

$$= \int\frac{dt}{\sqrt{t}} = 2\sqrt{t} = 2\sqrt{x^2+4x+10} + C_1 \qquad \text{...(2)}$$

Now,

$$I_2 = \int\frac{dx}{\sqrt{x^2+4x+10}} = \int\frac{dx}{\sqrt{(x+2)^2+(\sqrt{6})^2}}$$

$$= \log\left|(x+2)+\sqrt{(x+2)^2+6}\right| + C_2$$

from (1), (2) & (3),

$$\int \frac{5x+3}{\sqrt{x^2+4x+10}}dx = 5\sqrt{x^2+4x+10} - 7\log\left|(x+2)+\sqrt{x^2+4x+10}\right| + C$$

OR

Let $I = \int \frac{x-1}{(x-1)\sqrt{x^3+x^2+x}}dx.$

$\Rightarrow \quad I = \int \frac{x^2-1}{(x-1)^2\sqrt{x^3+x^2+x}}dx$ [Multiplying the N^+ and D^r by $(x + 1)$]

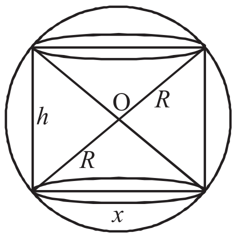
$\Rightarrow \quad I = \int \frac{(x^2-1)}{(x^2+2x+1)\sqrt{x^3+x^2+x}}dx$

$\Rightarrow \quad I = \int \frac{1-\frac{1}{x^2}}{\left(x+\frac{1}{x}+2\right)\sqrt{x+\frac{1}{x}+1}}dx$ [Dividing N^r and D^r by x^2]

Let $x+\frac{1}{x}+1 = t^2$. Then, $d\left(x+\frac{1}{x}+1\right) = d(t^2) \Rightarrow \left(1-\frac{1}{x^2}\right)dx = 2t\,dt$

$\Rightarrow \quad I = \int \frac{2t\,dt}{(t^2+1)\sqrt{t^2}} = 2\int \frac{1}{t^2+1}dt = 2\tan^{-1}(t)+C = 2\tan^{-1}\sqrt{x+\frac{1}{x}+1}+C$

16. Radius of the sphere = R, let h be the height and x be the diameter of the base of the inscribed cylinder.



Then $h^2 + x^2 = (2R)^2 \Rightarrow h^2 + x^2 = 4R^2$ ...(i)

Volume of the cylinder = π (radius)^2 × height $\Rightarrow V = \pi\left(\frac{x}{2}\right)^2 \times h$

$V = \frac{1}{4}\pi x^2 h \Rightarrow V = \frac{1}{4}\pi h(4R^2 - h^2)$ [from (i)]

$\Rightarrow V = \pi R^2 h - \frac{1}{4}\pi h^3$

$\therefore \frac{dV}{dh} = \pi R^2 - \frac{3}{4}\pi h^2 = \pi\left(R^2 - \frac{3}{4}h^2\right)$

$\Rightarrow \frac{dV}{dh} = 0 \Rightarrow R^2 = \frac{3}{4}h^2 \Rightarrow h = \frac{2R}{\sqrt{3}}$

Also, $\frac{d^2V}{dh^2} = -\frac{3}{4}\cdot 2\pi h = -\frac{3}{2}\pi h$

At $h = \frac{2R}{\sqrt{3}}$

$\frac{d^2V}{dh^2} = -\frac{3}{2}\pi\left(\frac{2R}{\sqrt{3}}\right) = -ve$

$\Rightarrow V$ is maximum at $h = \frac{2R}{\sqrt{3}}$

$\therefore$ Maximum volume at $h = \frac{2R}{\sqrt{3}}$

$= \frac{1}{4}\pi\left(\frac{2R}{\sqrt{3}}\right)\left(4R^2 - \frac{4R^2}{3}\right) = \frac{\pi R}{2\sqrt{3}}\left(\frac{8R^2}{3}\right) = \frac{4\pi R^3}{3\sqrt{3}}$ cube units

17. $f(x) = |x| - |x+1|$,

When $x < -1$, $f(x) = -x - [-(x+1)] = -x + x + 1 = 1$

When $-1 \le x < 0$, $f(x) = -x - (x+1) = -2x - 1$

When $x \ge 0$, $f(x) = x - (x+1) = -1$

$$\therefore f(x) = \begin{cases} 1 & , \text{if } x < -1 \\ -2x-1, & \text{if } -1 \le x < 0 \\ -1 & , \text{if } x \ge 0 \end{cases}$$

At $x = -1$, L.H.L. = $\lim\limits_{x \to (-1)^-} f(x) = \lim\limits_{x \to (-1)^-} (1) = 1$

R.H.L. = $\lim\limits_{x \to (-1)^+} f(x) = \lim\limits_{(x \to -1)^+} (-2x - 1) = 1$

$\therefore f(-1) = -2(-1) - 1 = 2 - 1 = 1$

$\therefore$ L.H.L.= R.H.L. = $f(-1)$ $\Rightarrow f$ is continuous at $x = -1$

At $x = 0$,

L.H.L. = $\lim\limits_{x \to 0^-} (-2x - 1) = -1$

$f(0) = -1$ (given)

R.H.L. = $\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+} (-1) = -1$

$\therefore$ L.H.L. = R.H.L. = $f(0)$

$\therefore f$ is continuous at $x = 0$

$\Rightarrow$ There is no point of discontinuity. Hence f is continuous for all $x \in \mathbf{R}$.

18. The differential equation is $\left(\frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}\right)\frac{dx}{dy} = 1$ or $\frac{dy}{dx} = \frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}$

or $\frac{dy}{dx} + \frac{1}{\sqrt{x}} y = \frac{e^{-2\sqrt{x}}}{\sqrt{x}}$

Linear equation of the form $\frac{dy}{dx}$ + Py = Q,

where P = $\frac{1}{\sqrt{x}}$, Q = $\frac{e^{2\sqrt{x}}}{\sqrt{x}}$

Now, $\int \frac{1}{\sqrt{x}}\, dx = \int x^{-1/2}\, dx = 2\sqrt{x}$

I.F. = $e^{\int p\,dx} = e^{2\sqrt{x}}$

The solution is $y \cdot e^{2\sqrt{x}} = \int \frac{e^{-2\sqrt{x}}}{\sqrt{x}} \times e^{2\sqrt{x}}\, dx + c = \int \frac{1}{\sqrt{x}}\, dx + c = 2\sqrt{x} + c$.

The required solution is $y\, e^{2\sqrt{x}} = 2\sqrt{x} + c$.

19. $f(x) = 2x^3 - 3x^2 - 36x + 7$

$f'(x) = 6x^2 - 6x - 36 = 6(x^2 - x - 6)$

$f'(x) = 6(x - 3)(x + 2)$

$\Rightarrow f'(x) = 0$ at $x = 3$ and $x = -2$

The points $x = 3$, $x = -2$, divide the real line into three disjoint intervals, viz, $(-\infty, -2)$, $(-2, 3)$, $(3, \infty)$.

Now $f'(x)$ is +ve in the intervals $(-\infty, -2)$ and $(3, \infty)$. Since in the interval $(-\infty, -2)$, each factor $(x - 3)$, $(x + 2)$ is –ve.

$\therefore f'(x) > 0$.

(a) f is strictly increasing in $(-\infty, -2) \cup (3, \infty)$

(b) In the interval $(-2, 3)$, $(x + 2)$ is +ve and $(x - 3)$ is –ve.

$f'(x) = 6(x - 3)(x + 2) = (+)(-) = $ –ve

$\therefore f$ is strictly decreasing in the interval $(-2, 3)$.

20. Shortest distance between the lines $\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$ and $\frac{x - x_2}{a_2} = \frac{y - y_2}{b_2} = \frac{z - z_2}{c_2}$ is

$$\frac{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_l & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2 + (a_1b_2 - a_2b_1)^2}}$$

The given lines are $\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1}$ and $\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1}$

These lines pass through the points (–1, –1, –1) and (3, 5, 7)

$x_2 - x_1 = 3 - (-1) = 4 \qquad y_2 - y_1 = 5 - (-1) = 6 \qquad z_2 - z_1 = 7 - (-1) = 8$

Now numerator $N = \begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = \begin{vmatrix} 4 & 6 & 0 \\ 7 & -6 & 1 \\ 1 & -2 & 1 \end{vmatrix}$

$\because$ a_1, b_1, c_1 are 7, –6, 1 and a_2, b_2, c_2 are 1, –2, 1

$\therefore$ $N = 4(-6+2) + 6(1-7) + 8(-14+6) = 4 \times (-4) + 6 \times (-6) + 8 \times (-8) = -16 - 36 - 64 = -116$

Denominator $D = \sqrt{(b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2 + (a_1b_2 - a_2b_1)^2}$

$\therefore$ $D = \sqrt{(-6+2)^2 + (1-7)^2 + (-14+6)^2}$

$= \sqrt{4^2 + (-6)^2 + (-8)^2} = \sqrt{16+36+64} = \sqrt{116}$

$\therefore$ Shortest distance $= \left|\frac{-116}{\sqrt{116}}\right| = \sqrt{116} = 2\sqrt{29}$.

21. Let $\vec{c} = x\hat{i} + y\hat{j} + z\hat{k}$

Also, $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{b} = \hat{j} - \hat{k}$

For $\vec{a} \times \vec{c} = \vec{b}$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ x & y & z \end{vmatrix} = \hat{j} - \hat{k}$$

$\Rightarrow$ $\hat{i}(z-y) - \hat{j}(z-x) + (y-x)\hat{k} = \hat{j} - \hat{k}$

$\therefore$ $z - y = 0$ ...(i)

$x - y = 1$ ...(ii)

$x - y = 1$ ...(iii)

Also, $\vec{a} \cdot \vec{c} = 3$

$\Rightarrow$ $x + y + z = 3$ ...(iv)

Solving equation we get $x = \frac{5}{3}, y = \frac{2}{3}$ and $z = \frac{2}{3}$

$\therefore$ $\vec{c} = \frac{5}{3}\hat{i} + \frac{2}{3}\hat{j} + \frac{2}{3}\hat{k} = \frac{1}{3}(5\hat{i} + 2\hat{j} + 2\hat{k})$

22. Let E_1, E_2 and E_3 be the events that boxes I, II and III are chosen, respectively.

Then $P(E_1) = P(E_2) = P(E_3) = \frac{1}{3}$

Also, let A be the event that 'the coin drawn is of gold'

Then $P(A|E_1) = P(\text{a gold coin from bag I}) = \frac{2}{2} = 1$

$P(A|E_2) = P(\text{a gold coin from bag II}) = 0$

$P(A|E_3) = P(\text{a gold coin from bag III}) = \frac{1}{2}$

Now, the probability that the other coin in the box is of gold

= the probability that gold coin is drawn from the box I.

$= P(E_1|A)$

By Bayes' theorem, we know that

$$P(E_1 | A) = \frac{P(E_1)P(A | E_1)}{P(E_1)P(A | E_1) + P(E_2)P(A | E_2) + P(E_3)P(A | E_3)}$$

$$= \frac{\frac{1}{3} \times 1}{\frac{1}{3} \times 1 + \frac{1}{3} \times 0 + \frac{1}{3} \times \frac{1}{2}} = \frac{2}{3}$$

23. Given equation can be written as $\tan^{-1}(1) - \tan^{-1} x = \frac{1}{2}\tan^{-1} x$

$\Rightarrow$ $\frac{3}{2}\tan^{-1} x = \frac{\pi}{4}$ or $\tan^{-1} x = \frac{\pi}{6}$

$\Rightarrow$ $x = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$

Section-D

24. Let y be an arbitrary element in range of f.

Let $y = 9x^2 + 6x - 5 = 9x^2 + 6x + 1 - 6 \Rightarrow y = (3x+1)^2 - 6$

$\Rightarrow y + 6 = (3x+1)^2 \Rightarrow 3x + 1 = \sqrt{y+6} \Rightarrow x = \dfrac{\sqrt{y+6}-1}{3} = g(y)$

Let g : Range $f \to \mathbf{R}_+$, such that $g(y) = \dfrac{\sqrt{y+6}-1}{3}$

$gof(x) = g(f(x)) = g(9x^2 + 6x - 5)$

$= g[(3x+1)^2 - 6] = \dfrac{\left(\sqrt{(3x+1)^2-6+6}\right)-1}{3} = \dfrac{3x+1-1}{3} = x$

$\Rightarrow gof(x) = x$ Now $fog(y) = f(g(y)) = f\left(\dfrac{\sqrt{y+6}-1}{3}\right)$

$= \left[3\left(\dfrac{\sqrt{y+6}-1}{3}\right)+1\right]^2 - 6 = y \Rightarrow fog(y) = y \Rightarrow gof = I_x, fog = I_y$

$\Rightarrow$ f is invertible $f^{-1}(y) = g(y) = \dfrac{\sqrt{y+6}-1}{3}$

OR

The set A $\therefore \{x \in Z : 0 \le x \le 12\} = \{0, 1, 2, ..., 12\}$

(i) $R = \{(a, b) : |a - b|$ is a multiple of $4\}$

$|a - b| = 4k \Rightarrow b = a + 4k$.

$\therefore R = \{(1, 5), (1, 9), (2, 6), (2, 10), (3, 7), (3, 11), (4, 8), (4, 12), (5, 9), (6, 10), (7, 11), (8, 12), (0, 0), (1, 1), (2, 2), ... (12, 12)\}$

(a) $(a - a) = 0 = 4k$, where $k = 0 \Rightarrow (a, a) \in R$. $\therefore$ R is reflexive.

(b) If $|a - b| = 4k$, then $|b - a| = 4k$

i.e. (a, b) and (b, a) both belong to R $\Rightarrow$ R is symmetric.

(c) $a - c = a - b + b - c$

When $a - b$ and $b - c$ are both multiples of 4 then $a - c$ is also a multiple of 4. This shows if (a, b) $(b, c) \in R$ then $a - c$ also $\in R$.

$\therefore$ R is an equivalence relation. The sets related to 1 are $\{(1, 5), (1, 9)\}$.

25. We have, $A = \begin{bmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix}$

$\therefore \quad BA = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{bmatrix} = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix} = 6I$

$\therefore \quad B^{-1} = \dfrac{A}{6} = \dfrac{1}{6}\begin{bmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{bmatrix}$ ...(i)

Given system of equation is:

$x - y = 3, 2x + 3y + 4z = 17$ and $y + 2z = 7$

or $\begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 17 \\ 7 \end{bmatrix}$

$\therefore \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix}^{-1}\begin{bmatrix} 3 \\ 17 \\ 7 \end{bmatrix} = \dfrac{1}{6}\begin{bmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{bmatrix}\begin{bmatrix} 3 \\ 17 \\ 7 \end{bmatrix} = \dfrac{1}{6}\begin{bmatrix} 6+34-28 \\ -12+34-28 \\ 6-17+35 \end{bmatrix} = \dfrac{1}{6}\begin{bmatrix} 12 \\ -6 \\ 24 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$

$\therefore \quad x = 2, y = -1$ and $z = 4$

OR

We may write the given matrix is

A = IA

$$\therefore \begin{bmatrix} 2 & -1 & 3 \\ -5 & 3 & 1 \\ -3 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

[Applying $R_2 \to R_2 + R_1$]

$$\Rightarrow \begin{bmatrix} 2 & -1 & 3 \\ -3 & 2 & 4 \\ -3 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

[Applying $R_3 \to R_3 - R_2$]

$$\Rightarrow \begin{bmatrix} 2 & -1 & 3 \\ -3 & 2 & 4 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} A$$

[Applying $R_1 \to R_1 + R_2$]

$$\Rightarrow \begin{bmatrix} -1 & 1 & 7 \\ -3 & 2 & 4 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} A$$

[Applying $R_2 \to R_2 - 3R_1$]

$$\Rightarrow \begin{bmatrix} -1 & 1 & 7 \\ 0 & -1 & -17 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ -5 & -2 & 0 \\ -1 & -1 & 1 \end{bmatrix} A$$

[Applying $R_1 \to R_1 + R_2$ and $R_3 \to (-1)\cdot R_3$]

$$\Rightarrow \begin{bmatrix} -1 & 0 & -10 \\ 0 & -1 & -17 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -3 & -1 & 0 \\ -5 & -2 & 0 \\ 1 & 1 & -1 \end{bmatrix} A$$

[Applying $R_1 \to R_1 + 10R_3$ and $R_2 \to R_2 + 17R_3$]

$$\Rightarrow \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 9 & -10 \\ 12 & 15 & -17 \\ 1 & 1 & -1 \end{bmatrix}$$

Taking '–1' common from R_1 and R_2

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -7 & -9 & 10 \\ -12 & -15 & 17 \\ 1 & 1 & -1 \end{bmatrix} A$$

So, the inverse of A is $\begin{bmatrix} -7 & -9 & 10 \\ -12 & -15 & 17 \\ 1 & 1 & -1 \end{bmatrix}$

26. $\int_0^{\pi} \frac{(\pi - x)\tan(\pi - x)}{\sec(\pi - x) + \tan(\pi - x)} dx \qquad \left[\because \int_0^a f(x)dx = \int_0^a f(a-x)\, dx\right]$

$$= \int_0^{\pi} \frac{-(\pi - x)\tan x}{-(\sec x + \tan x)} dx$$

$$I = \int_0^{\pi} \frac{(\pi - x)\tan x}{(\sec x + \tan x)} dx \qquad \text{...(ii)} \quad \text{Adding (i) and (ii)}$$

$$2I = \int_0^{\pi} \frac{x\tan x + (\pi - x)\tan x}{(\sec x + \tan x)} dx$$

$$= \int_0^{\pi} \frac{x\tan x + \pi\tan x - x\tan x}{(\sec x + \tan x)} dx$$

$$= \int_0^{\pi} \frac{\tan x}{\sec x + \tan x} dx = \pi \int_0^{\pi} \frac{\frac{\sin x}{\cos x}}{\frac{1}{\cos x} + \frac{\sin x}{\cos x}} dx$$

$$= \int_0^{\pi} \frac{\sin x + 1 - 1}{1 + \sin x} dx = \pi \int_0^{\pi} \frac{1 + \sin x - 1}{1 + \sin x} dx$$

$$= \pi = \int_0^{\pi} 1\, dx - \pi \int_0^{\pi} \frac{dx}{1 + \sin x}$$

$= \pi[x]_0^{\pi} - \pi \int_0^{\pi} \frac{(1-\sin x)}{(1+\sin x)(1-\sin x)} dx$

$= \pi^2 - \pi \int_0^{\pi} \frac{1-\sin x}{1-\sin^2 x} dx$

$= \pi^2 - \pi \int_0^{\pi} \frac{1-\sin x}{\cos^2 x} dx$

$= \pi^2 - \pi [\tan x - \sec x]_0^{\pi}$

$= \pi^2 - \pi [0 - (-1 - 1)]$

$= \pi^2 - 2\pi,$

$I = \frac{\pi^2 - 2\pi}{2} = \frac{\pi}{2}(\pi - 2)$

OR

Here, $a = 1$, $b = 3$ and $f(x) = e^{2-3x} + x^2 + 1$. Therefore, $h = \frac{3-1}{n} = \frac{2}{n}$ and $nh = 2$.
Substituting these values in

$\int_a^b f(x)dx = \lim_{h \to 0} h[f(a) + f(a+h) + f(a+2h) + \ldots + f(a+(n-1)h]$, we obtain

$I = \int_1^3 (e^{2-3x} + x^2 + 1)dx = \lim_{h \to 0} h[f(1) + f(1+h) + f(1+2h) + \ldots + f(1+(n-1)h)]$

$\Rightarrow \quad I = \lim_{h \to 0} h\left[(e^{2-3\times 1} + 1^2 + 1) + \left\{e^{2-3(1+h)} + (1+h)^2 + 1\right\} + (e^{2-3(1+2h)} + (1+2h)^2 + 1\} + \ldots + \{e^{2-3\{1+(n-1)h\}} + \{1+(n-1)h\}^2 + 1\right]$

$\Rightarrow \quad I = \lim_{h \to 0} h\left[\left\{e^{-1} + e^{-1-3h} + e^{-1-3(2h)} + \ldots + e^{-1-3(n-1)h}\right\} + \left\{1^2 + (1+h)^2 + (1+2h)^2 + (1+3h)^2 + \ldots + \{1+(n-1)h^2\} + n\right\}\right]$

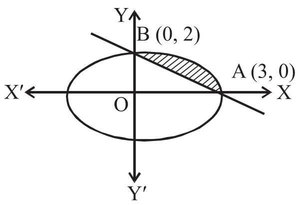
$\Rightarrow \quad I = \lim_{h \to 0} h\left[e^{-1}\left\{1 + e^{-3h} + e^{-3(2h)} + e^{-3(3h)} + \ldots e^{-3(n-1)h}\right\} + \left\{n + 2(1+2+3+\ldots+(n-1))\right\} + h^2\left\{1^2 + 2^2 + 3^2 + \ldots + (n-1)^2\right\} + n\right]$

$\Rightarrow \quad I = \lim_{h \to 0} h\left[e^{-1}\left\{\frac{(e^{-3h})^n - 1}{e^{-3h} - 1}\right\} + \left\{2n + 2h\frac{n(n-1)}{2} + \frac{h^2}{6}n(n-1)(2n-1)\right\}\right]$

$\Rightarrow \quad I = \lim_{h \to 0}\left[he^{-1}\left(\frac{e^{-3ng} - 1}{e^{-3h} - 1}\right) + 2nh + nh(nh-h) + \frac{1}{6}nh(nh-h)(2nh-h)\right]$

$\Rightarrow \quad I = \lim_{h \to 0}\left[e^{-1} \times \frac{(e^{-6} - 1)}{-3\left(\frac{e^{-3h} - 1}{-3h}\right)} + 4 + 2(2-h) + \frac{2}{6}(2-h)(4-h)\right] \qquad [\because nh = 12]$

27. The given ellipse is $\frac{x^2}{9} + \frac{y^2}{4} = 1 \quad \Rightarrow y^2 = \frac{4}{9}(9 - x^2) \quad \Rightarrow y = \frac{2}{3}\sqrt{9 - x^2}$



It is an ellipse with vertices at A (3, 0) and B (0, 2) and length of the major axis = 2 (3) = 6 and length of the minor axis = 2 (2) = 4

Line $\frac{x}{3} + \frac{y}{2} = 1 \Rightarrow y = \left(\frac{6-2x}{3}\right)$.
It is a straight line passing through A (3, 0) and B (0, 2).

Smaller area common to both is shaded.

Shaded Area $= \frac{2}{3}\int_0^3 \sqrt{9-x^2}\,dx - \int_0^3 \left(\frac{6-2x}{3}\right)dx = \frac{2}{3}I_1 - \frac{1}{3}I_2$

where $I_1 = \int_0^3 \sqrt{9-x^2}\,dx$ and $I_2 = \int_0^3 (6-2x)\,dx$.

For I_1, put $x = 3 \sin \theta$ so that $dx = 3 \cos \theta \, d\theta$.

when $x = 0$, $\theta = 0$ and when $x = 3$, $\theta = \frac{\pi}{2}$.

$\therefore I_1 = \int_0^{\frac{\pi}{2}} \sqrt{9 - 9\sin^2\theta} \cdot 3\cos\theta \, d\theta$

$= 9\int_0^{\frac{\pi}{2}} \sqrt{1-\sin^2\theta} \cdot \cos\theta \, d\theta = 9\int_0^{\frac{\pi}{2}} \cos^2\theta d\theta = \frac{9}{2}\int_0^{\frac{\pi}{2}} (1+\cos 2\theta) d\theta$

$= \frac{9}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{2}} = \frac{9}{2}\left(\frac{\pi}{2} - 0\right) = \frac{9\pi}{4}$

$I_2 = \int_0^3 (6-2x)dx = [6x - x^2]_0^3 = 18 - 9 = 9$

Shaded area $= \frac{2}{3} \times \frac{9\pi}{4} - \frac{1}{3} \times 9 = \frac{3\pi}{2} - 3 = \frac{3}{2}(\pi - 2)$ sq. units.

28. The two given planes are

$\vec{r} \cdot (\hat{i} + 2\hat{j} + 3\hat{k}) - 4 = 0$ ...(i)

$\vec{r} \cdot (2\hat{i} + \hat{j} - \hat{k}) + 5 = 0$ ...(ii)

A plane which contains the line of intersection of the planes (i) and (ii) is

$\vec{r} \cdot (\hat{i} + 2\hat{j} + 3\hat{k}) - 4 + \lambda[\vec{r} \cdot (2\hat{i} + \hat{j} - \hat{k}) + 5] = 0$

or $\vec{r} \cdot [(1+2\lambda)\hat{i} + (2+\lambda)\hat{j} + (3-\lambda)\hat{k}] - 4 + 5\lambda = 0$ ...(iii)

Now the plane (iii) is perpendicular to the plane

$\vec{r} \cdot (5\hat{i} + 3\hat{j} - 6\hat{k}) + 8 = 0$ ...(iv)

$\Rightarrow (1 + 2\lambda)\, 5 + (2 + \lambda)\, 3 + (3 - \lambda)(-6) = 0$

or $5 + 10\lambda + 6 + 3\lambda - 18 + 6\lambda = 0$

$\Rightarrow 19\lambda - 7 = 0 \quad \Rightarrow \lambda = \frac{7}{19}$

Putting the value of λ in (iii)

$\vec{r}\left[\left(1 + \frac{14}{19}\right)\hat{i} + \left(2 + \frac{7}{19}\right)\hat{j} + \left(3 - \frac{7}{19}\right)\hat{k}\right] - 4 + 5\frac{7}{19} = 0$

$= \vec{r}\left[\frac{33}{19}\hat{i} + \frac{45}{19}\hat{j} + \frac{50}{19}\hat{k}\right] + \frac{-76+35}{19} = 0$

or $\vec{r} \cdot (33\hat{i} + 45\hat{j} + 50\hat{k}) - 41 = 0$

This is the required plane.

29. Let the executive class air tickets and economy class tickets sold be x and y.

As the seating capacity of the aeroplane is 200, $x + y \leq 200$.

As 20 tickets for the executive class are to be reserved we have $x \geq 20$.

And as the number of tickets of the economy class should be at least 4 times that of the executive class $y \geq 4x$. Profit on the sale of x tickets of the executive class and y tickets of the economy class $Z = 1{,}000x + 600y$.

$\therefore$ LPP is to maximized,

$Z = 1{,}000x + 600y$ subject of constraints of $x + y \leq 200$, $x \geq 20$, $y \geq 4x$ and $x, y \geq 0$.

The region satisfying the inequality $x + y \leq 200$, $x \geq 20$ and $y \geq 4x$ is ABC.

$Z = 1{,}000x + 600y$

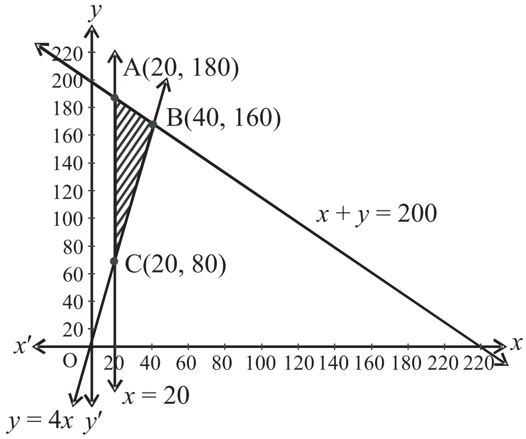
At A (20, 180), $Z = 1{,}000 \times 20 + 600 \times 180 = 20{,}000 + 1{,}08{,}000 = 1{,}28{,}000$

At B (40, 160), $Z = 1{,}000 \times 40 + 600 \times 160 = 40{,}000 + 96{,}000 = 1{,}36{,}000$

At C (20, 80), $Z = 1{,}000 \times 20 + 600 \times 80 = 20{,}000 + 48{,}000 = 68{,}000$

$\Rightarrow$ Z is maximum when $x = 40$, $y = 160$

$\Rightarrow$ To get maximum profit of ₹13,600, 40 tickets of the executive class and 160 tickets of the economy class should be sold.


y
220
200
180
160
140
120
100
80
60
40
20
A(20, 180)
B(40, 160)
C(20, 80)
x + y = 200
x'
x
O
20 40 60 80 100 120 140 160 180 220 220
x = 20
y = 4x
y'

❖❖❖❖❖