STUDY mate

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Hints/Solutions to Studymate Practice Boards Paper Mathematics (Class – XII)

Code No. 41/1

1. As the given vectors are coplanar

$$\begin{vmatrix} 1 & -1 & 1 \\ 3 & -1 & 2 \\ 1 & \lambda & -3 \end{vmatrix} = 0$$

$1[3 - 2\lambda] + 1[-9 - 2] + 1[3\lambda + 1] = 0$

$3 - 2\lambda - 11 + 3\lambda + 1 = 0$

$\lambda = 7$

$\therefore \quad \lambda^2 = 49$ **[1]**

2. $|\text{Adj } A| = 16$ **[1]**

$\Rightarrow \quad |A|^{n-1} = 4^2$

$\Rightarrow \quad n = 3$

3. $[2] = \{x : x \in A, 2x \text{ is a perfect square}\}$ **[1]**

$= \{2, 8, 18, 32, 50, 72, 98\}$

4. 16 **[1]**

5. Let $\sin^{-1}(0.4) = \theta$

$\sin(3\sin^{-1} 0.4) = \sin 3\theta = 3\sin\theta - 4\sin^3\theta$ **[1]**

$= 3(0.4) - 4(0.4)^3 = 1.2 - 2.256 = 0.944$ **[1]**

(where, $\theta = \sin^{-1}(0.4) \Rightarrow \sin\theta = 0.4$)

6. From the definition of a matrix,

$(AB)(AB)^{-1} = I$ **[½]**

Pre multiplying both sides by A^{-1}

$A^{-1}(AB)(AB)^{-1} = A^{-1}I$

$(A^{-1}A)B(AB)^{-1} = A^{-1}$ $\quad [A^{-1}I = A^{-1}]$

$IB(AB)^{-1} = A^{-1}$

$B(AB)^{-1} = A^{-1}$ **[1]**

Pre multiplying both sides by B^{-1} **[½]**

$B^{-1}B(AB)^{-1} = B^{-1}A^{-1}$

$I(AB)^{-1} = B^{-1}A^{-1}$

$(AB)^{-1} = B^{-1}A^{-1}$

7. Let $\sin^{-1}\frac{1}{3} = A$

$\Rightarrow \quad \sin A = \frac{1}{3}$ **[½]**

$\Rightarrow \quad \cos A = \frac{2\sqrt{2}}{3}$ **[½]**

$\Rightarrow \quad A = \cos^{-1}\left(\frac{2\sqrt{2}}{3}\right)$

$\therefore \quad \sin^{-1}\left(\frac{1}{3}\right) = \cos^{-1}\frac{2\sqrt{2}}{3}$

$\Rightarrow \quad \lambda = \frac{2\sqrt{2}}{3}$ **[½]**

Now, $\sin^{-1}\frac{2\sqrt{2}}{3} + \sin^{-1}\frac{1}{3} = \sin^{-1}\frac{2\sqrt{2}}{3} + \cos^{-1}\frac{2\sqrt{2}}{3} = \frac{\pi}{2}.$ **[½]**

$\left[\because \quad \sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}.\right]$

8. $y = 5x - 2x^3$

Let slope of curve is m

Slope of curve i.e. $\frac{dy}{dx} = 5 - 6x^2$ **[½]**

$\therefore \quad m = 5 - 6x^2$

Rate of change of slope

$\frac{dm}{dt} = -12x.\frac{dx}{dt}$

$\frac{dm}{dt} = -12x.2 \qquad \left[\text{Given } \frac{dx}{dt} = 2\right]$

$\frac{dm}{dt} = -24x$ **[1]**

$\left.\frac{dm}{dt}\right|_{x=3} = -72$ units/sec. **[½]**

9. Here, $f(x) = \log\left(\frac{2-x}{2+x}\right)$

$f(-x) = \log\left(\frac{2+x}{2-x}\right) = \log\left(\left(\frac{2-x}{2+x}\right)^{-1}\right)$

$= -\log\left(\frac{2-x}{2+x}\right) = -f(x)$ **[1]**

$\Rightarrow \quad f$ is an odd function, therefore using

$\int_{-a}^{a} f(x)dx = 0, f(x) = -f(x)$ **[½]**

We have $\int_{-1}^{1} \log\left(\frac{2-x}{2+x}\right)dx = 0$ **[½]**

10. As per question.

$\frac{dy}{dx} = (2x)\ (3y)$ **[½]**

$\frac{dy}{y} = 6x\ dx$

Integrating both sides

$\int \frac{dy}{y} = 6\int x\, dx$ **[½]**

$\log y = 3x^2 + C$, is the general solution of curve.

The curve passes through (0, 1)

$\therefore \quad \log 1 = C$

$\Rightarrow$ $C = 0$ **[½]**

$\therefore$ $\log y = 3x^2$ is the particular solution of the required curve.

$\therefore$ the equation of curve is $y = e^{3x^2}$. **[½]**

11. Let $\vec{a}, \vec{b}$ be representing adjacent sides of the parallelogram, therefore,

$$\vec{d}_1 = \vec{a} + \vec{b}$$

and $\vec{d}_2 = \vec{a} - \vec{b}$

$\Rightarrow$ $\vec{a} = \frac{1}{2}(\vec{d}_1 + \vec{d}_2)$ and $\vec{b} = \frac{1}{2}(\vec{d}_1 - \vec{d}_2)$ **[½]**

Now, Area of the parallelogram

$= |\vec{a} \times \vec{b}|$ **[½]**

$$= \left|\frac{1}{2}(\vec{d_1} + \vec{d_2}) \times \frac{1}{2}(\vec{d_1} - \vec{d_2})\right|$$

$$= \left|\frac{1}{4}(\vec{d_1} \times \vec{d_1} - \vec{d_1} \times \vec{d_2} + \vec{d_2} \times \vec{d_1} - \vec{d_2} \times \vec{d_2})\right|$$

$$= \left|\frac{1}{4}(-\vec{d_1} \times \vec{d_2} - \vec{d_1} \times \vec{d_2})\right|$$

$= \left|\frac{-2}{4}(\vec{d_1} \times \vec{d_2})\right| = \frac{|\vec{d_1} \times \vec{d_2}|}{2}$ **[1]**

12. Let E, F and G be the events representing solving the problem by student 1st, solving the problem by student 2nd and solving the problem by student 3rd respectively.

Here, we have

$$P(E') = \frac{1}{2}, P(F') = \frac{2}{3} \text{ and } P(G') = \frac{3}{4}$$

= P (atleast one of then solve the problem) **[1]**

= 1 – P(none of them solve the problem)

$= 1 - \frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} = \frac{3}{4}$ **[1]**

$\therefore$ Required probability is 3/4.

13. Here, $A = \begin{vmatrix} x & x^2 & 1+bx^3 \\ y & y^2 & 1+py^3 \\ z & z^2 & 1+pz \end{vmatrix}$

$= \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} + \begin{vmatrix} x & x^2 & px^3 \\ y & y^2 & py^3 \\ z & z^2 & pz^3 \end{vmatrix}$ **[½]**

Taking common x, y, z, p from R_1, R_2, and C_3 respectively from 2nd determinant,

$= \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} + pxyz\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$ **[½]**

Applying $C_1 \leftrightarrow C_2$ in 1st determinant and $C_1 \leftrightarrow C_3$ in 2nd determinant

$$= -\begin{vmatrix} x^2 & x & 1 \\ y^2 & y & 1 \\ z^2 & z & 1 \end{vmatrix} - (pxyz)\begin{vmatrix} x^2 & x & 1 \\ y^2 & y & 1 \\ z^2 & z & 1 \end{vmatrix}$$

$$= -(1+pxyz)\begin{vmatrix} x^2 & x & 1 \\ y^2 & y & 1 \\ z^2 & z & 1 \end{vmatrix}$$ **[½]**

Applying $R_1 \to R_1 - R_3$, $R_2 \to R_2 - R_3$ **[½]**

$$= -(1+pxyz)\begin{vmatrix} (x-z)(x+z) & x-z & 0 \\ (y-z)(y+z) & y-z & 0 \\ z^2 & z & 1 \end{vmatrix}$$

Taking common $(x - z)$ and $(y - z)$ from R_1 and R_2 respectively

$$= -(1+pxyz)(x-z)(y-z)\begin{vmatrix} x+z & 1 & 0 \\ y+z & 1 & 0 \\ z^2 & z & 1 \end{vmatrix}$$ **[½]**

Expanding along C_3

$= -(1 + pxyz)(x - z)(y - z)(x - y)$ **[½]**

$\therefore \quad \Delta = -(1 + pxyz)(x - z)(y - z)(x - y)$

$= (x - y)(y - z)(z - x)(1 + pxyz)$

Hence, the required four conditions which make $\Delta = 0$, **[1]**

are, either $x = y$ or $y = z$ or $z = x$ or $pxyz = -1$

14. Here, $f(x) = \begin{cases} x^2 + 3x + p, & x \le 1 \\ qx + 2, & x > 1 \end{cases}$ is differentiable

at $x = 1$, therefore it will also be continuous at $x = 1$.

Continuity:

LHL $= \lim_{x\to 1^-}(x^2 + 3x + p) = 4 + p$

RHL $= \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}(qx + 2) = q + 2$

and $f(1) = 4 + p$

$\Rightarrow \quad 4 + p = q + 2$

or $\quad p - q + 2 = 0 \quad \ldots$(i) **[1]**

Differentiability:

$$L\,f'(1) = \lim_{h\to 0}\frac{f(1-h) - f(1)}{-h} = \lim_{h\to 0}\frac{[(1-h)^2 + 3(1-h) + p] - (4+p)}{-h}$$

$$= \lim_{h\to 0}\frac{h^2 - 5h}{-h} = 5$$ **[1]**

$$R\,f'(1) = \lim_{h\to 0}\frac{f(1+h) - f(1)}{h} = \lim_{h\to 0}\frac{[q(1+h) + 2] - [4+p]}{h}$$

$$= \lim_{h\to 0}\frac{q + qh - 2 - p}{h} \quad \text{(from (i))}$$ **[1]**

$$= \lim_{h\to 0}\frac{qh}{h} = q$$

Now, $Lf'(1) = Rf'(1)$

$\Rightarrow \quad q = 5$ and $p = 3$ **[1]**

OR

At $x = 0$

LHL $= \lim_{x\to 0^-}\left(\frac{\log(1+ax) - \log(1+3x)}{x}\right) = \lim_{x\to 0^-}\left[a\,\frac{\log(1+ax)}{ax} - 3\,\frac{\log(1+3x)}{3x}\right]$

$$= a\lim_{x\to 0^-}\left[\frac{\log(1+ax)}{ax}\right] - 3\lim_{x\to 0^-}\left[\frac{\log(1+3x)}{3x}\right] \qquad \left[\because \quad \lim_{x\to 0}\frac{\log(1+x)}{x}=1\right]$$ **[2]**

$\Rightarrow \quad a - 3$

$$\text{RHL} = \lim_{x\to 0^+}\left(\frac{\sin^2 bx}{x^2}\right) = \lim_{x\to 0^+}\left(\frac{\sin bx}{bx}\right)^2 b^2 = b^2$$ **[1]**

$f(0) = 1$

$\therefore \quad f$ is continuous at $x = 0$, therefore $a - 3 = 1$ and $b^2 = 1$

$\Rightarrow \quad a = 4,\ b = \pm 1$ **[1]**

15. $\sqrt{y} = \tan^{-1} x$

$\Rightarrow \quad y = (\tan^{-1} x)^2$ **[½]**

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2\tan^{-1}x}{1+x^2}$$ **[1]**

$$\text{or} \quad \frac{dy}{dx}(1+x^2) = 2\tan^{-1}x$$ **[½]**

$$\Rightarrow \quad \frac{d^2y}{dx^2}(1+x^2) + 2x\frac{dy}{dx} = \frac{2}{1+x^2}$$ **[1]**

$$\text{i.e.,} \quad (x^2+1)^2\frac{d^2y}{dx^2} + 2x(x^2+1)\frac{dy}{dx} = 2$$ **[1]**

16. $$f(x) = \frac{4\sin x - 2x - x\cos x}{2+\cos x}$$

$$= \frac{4\sin x}{2+\cos x} - \frac{x(2+\cos x)}{2+\cos x}$$

$$= \frac{4\sin x}{2+\cos x} - x$$ **[½]**

$$f'(x) = \frac{4\cos x(2+\cos x) - (0-\sin x)4\sin x}{(2+\cos x)^2} - 1$$

$$= \frac{8\cos x + 4(\cos^2 x + \sin^2 x)}{(2+\cos x)^2} - 1$$

$$= \frac{8\cos x + 4 - 4 - \cos^2 x - 4\cos x}{(2+\cos x)^2}$$ **[1]**

$$= \frac{4\cos x - \cos^2 x}{(2+\cos x)^2}$$

$$f'(x) = \frac{(4-\cos x)\cos x}{(2+\cos x)^2}$$

Putting $f'(x) = 0$ gives $\cos x = 0 \quad \Rightarrow \quad x = \frac{\pi}{2}, \frac{3\pi}{2} \quad [\because x \in [0, 2\pi]]$

Interval	(4 – cos x)	cos x	(2 + cos x)²	f'(x)	f(x)
$\left[0, \frac{\pi}{2}\right)$	(+)	(+)	(+)	(+) (+) (+) = (+)	↑
$\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$	(+)	(–)	(+)	(+) (–) (+) = (—)	↓
$\left(\frac{3\pi}{2}, 2\pi\right]$	(+)	(+)	(+)	(+) (+) (+) = (+)	↑

[2]

(i) Increasing: $x \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{3\pi}{2}, 2\pi\right]$

(ii) Decreasing: $x \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ **[½]**

OR

Here, curve is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{...(i)}$$

$\Rightarrow \quad y' = \frac{-b^2}{a^2}\frac{x}{y}$ **[½]**

Let (x_1, y_1) be the point where $x \cos \alpha + y \sin \alpha = p$ touches the curve (i).

Now, slope of the line $x \cos \alpha + y \sin \alpha = p$, is

$= - \cot \alpha$ **[½]**

Also, using curve (i), slope of the same line at (x_1, y_1) is

$$= \frac{-b^2}{a^2}\frac{x_1}{y_1}$$

$\therefore \quad = \frac{b^2}{a^2}\frac{x_1}{y_1} = \cot \alpha \quad$ ...(ii) **[½]**

Also, pt.(x_1, y_1) lies on $x \cos \alpha + y \sin \alpha = p$

$\therefore \quad x_1 \cos \alpha + y_1 \sin \alpha = p \quad$ ...(iii)

Solving (ii) and (iii), we get

$x_1 = \frac{pa^2 \cos \alpha}{a^2 \cos^2 \alpha + b^2 \sin^2 \alpha}$ and $y_1 = \frac{pb^2 \sin \alpha}{a^2 \cos^2 \alpha + b^2 \sin^2 \alpha}$ **[½]**

Now, put values of x_1 and y_1 in (i), we get

$$\frac{p^2 a^4 \cos^2 \alpha}{a^2(a^2 \cos^2 \alpha + b^2 \sin^2 \alpha)^2} + \frac{p^2 b^4 \sin^2 \alpha}{b^2(a^2 \cos^2 \alpha + b^2 \sin^2 \alpha)^2} = 1$$

$$p^2(a^2 \cos^2 \alpha + b^2 \sin^2 \alpha) = (a^2 \cos^2 \alpha + b^2 \sin 2\alpha)^2$$

i.e., $a^2 \cos^2 \alpha + b^2 \sin^2 \alpha = p^2$

Hence proved. **[2]**

17. Let x be the increase in subscription charges, then according to the question,

Revenue, $R = (500 - x)(300 + x)$

$= 150000 + 200x - x^2$ **[2]**

$\Rightarrow \quad \frac{dR}{dx} = 200 - 2x$ **[½]**

$\frac{dR}{dx} = 0$

$\Rightarrow \quad 200 - 2x = 0$

$\Rightarrow \quad x = 100$

Now $\left.\frac{d^2R}{dx^2}\right|_{x=100} = -2 < 0$

$\therefore \quad$ R is maximum at $x = 100$ **[1]**

Therefore, Maximum revenue will be (500 – 100) (300 + 100) = Rs. 1,60,000. **[½]**

18. $I = \int \frac{1}{\cos(x+a)\cos(x+b)}\,dx$

$= \frac{1}{\sin(a-b)}\int \frac{\sin(a-b)}{\cos(x+a)\cos(x+b)}\,dx$ **[1]**

$= \frac{1}{\sin(a-b)}\int \frac{\sin[(x+a)-(x+b)]}{\cos(x+a)\cos(x+b)}\,dx$

$= \frac{1}{\sin(a-b)}\int \frac{\sin(x+a).\cos(x+b)-\cos(x+a)\sin(x+b)}{\cos(x+a)\cos(x+b)}\,dx$ **[1]**

$= \frac{1}{\sin(a-b)}\int \left(\frac{\sin(x+a)}{\cos(x+a)} - \frac{\sin(x+b)}{\cos(x+b)}\right) dx$

$= \frac{1}{\sin(a-b)}\int \left(\tan(x+a) - \tan(x+b)\right) dx$

$= \frac{1}{\sin(a-b)}\left[(\log|\sec(x+a)| - \log\sec|x+b|)\right] + C$ **[2]**

$= \frac{1}{\sin(a-b)}\log\left|\frac{\sec(x+a)}{\sec(x+b)}\right| + C$

19. Given differential equation of the form

$\frac{dy}{dx} + Py = Q,$

Therefore, $I.F = e^{\int -1\,dx} = e^{-x}$ **[1]**

Multiplying both sides of equation by $I.F$, we get

$e^{-x}\frac{dy}{dx} - e^{-x}y = e^{-x}\cos x$

or $I = \frac{d}{dx}\left(y\,e^{-x}\right) = e^{-x}\cos x$

On integrating both sides with respect to x, we get **[½]**

$\therefore$ $ye^{-x} = \int e^{-x}\cos x\,dx + C$

Let $I = \int e^{-x}\cos x\,dx$

$= \cos x\left(-e^{-x}\right) - \int(-\sin x)(-e^{-x})\,dx$

$= -e^{-x}\cos x - \int e^{-x}\sin x\,dx$

$= -e^{-x}\cos x - \left[\sin x(-e^{-x}) - \int \cos x(-e^{-x})dx\right]$

$= -e^{-x}\cos x + e^{-x}\sin x - \int e^{-x}\cos x dx$

or $I = -\,e^{-x}\cos x + \sin x\, e^{-x} - I$ **[2]**

or $2I = (\sin x - \cos x)\,e^{-x}$

or $I = \frac{(\sin x - \cos x)e^{-x}}{2}$

Substituting the value of I in equation (1), we get

$ye^{-x} = \left(\frac{\sin x - \cos x}{2}\right)e^{-x} + C$

or $y = \left(\frac{\sin x - \cos x}{2}\right) + Ce^{x}$ **[½]**

which is the general solution of the given differential equation.

OR

$$\left(\frac{2+\sin x}{1+y}\right)\frac{dy}{dx}=-\cos x$$

$\Rightarrow \quad \int \frac{1}{1+y}\,dy=-\int \frac{\cos x}{2+\sin x}\,dx$

$\Rightarrow \quad \log|1+y| = -\log|2+\sin x| + C$ **[1]**

$(1+y)(2+\sin x) = C$ **[½]**

Put $y = 1$ and $x = 0$, we get, $C = 4$ **[½]**

$\therefore \quad (1+y)(2+\sin x) = 4$

or $\quad 1+y=\frac{4}{2+\sin x}$

i.e., $\quad y=\frac{2-\sin x}{2+\sin x}$

Hence, $y(x)=\frac{2-\sin x}{2+\sin x}$ **[½]**

and $\quad y\left(\frac{\pi}{2}\right)=\frac{1}{3}.$ **[½]**

20. Here,

$\vec{c}=\lambda(\vec{a}\times\vec{b})$

$\Rightarrow \quad \vec{c}$ and $\vec{a}\times\vec{b}$ are collinear vector,

$\vec{c}$ is perpendicular to $\vec{a}$ and $\vec{b}$ vectors.

i.e., $\quad \vec{c}\,.\,\vec{a} = \vec{c}\,.\,\vec{b}=0$ ...(i) **[1]**

Now, $|\vec{a}+\vec{b}+\vec{c}|=1$

$\Rightarrow \quad |\vec{a}+\vec{b}+\vec{c}|^2=1$ **[½]**

$\Rightarrow \quad (\vec{a}+\vec{b}+\vec{c}).(\vec{a}+\vec{b}+\vec{c})=1$ **[½]**

$\Rightarrow \quad |\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2+2\vec{a}\,.\,\vec{b}+2\vec{b}\,.\,\vec{c}+2\vec{a}\,.\,\vec{c}=1$

or $\quad \frac{1}{2}+\frac{1}{3}+\frac{1}{6}+2\vec{a}\,.\,\vec{b} = 1$ **[½]**

(from (i) and $|\vec{a}|=\frac{1}{\sqrt{2}},|\vec{b}|=\frac{1}{\sqrt{3}}$ and $|\vec{c}|=\frac{1}{\sqrt{6}}$) **[½]**

$\Rightarrow \quad \vec{a}\,.\,\vec{b}=0$

$\Rightarrow \quad \vec{a}$ and $\vec{b}$ are perpendicular

$\therefore \quad$ Angle between $\vec{a}$ and $\vec{b}$ is 90. **[1]**

21. Given planes are

$\vec{r}\,.\,(\hat{i}+3\hat{j})-6=0$ ...(i)

$\vec{r}\,.\,(3\hat{i}-\hat{j}-4\hat{k})=0$ ...(i)

Now, equation of the plane, passing through intersection of planes (i) and (ii) is,

$\vec{r}\,.\,((\hat{i}+3\hat{j})+\lambda(3\hat{i}-\hat{j}-4\hat{k})=6$

or $\quad \vec{r}\,.\,((1+3\lambda)\hat{i}+(3-\lambda)\hat{j}-4\lambda\hat{k})=6$ ...(iii) **[1]**

$\because$ Perpendicular distance from origin of plane (iii) = 1

$\therefore$ $$\left|\frac{6}{\sqrt{(1+3\lambda)^2+(3-\lambda)^2+(-4\lambda)^2}}\right|=1$$ **[1]**

$6^2 = 26\lambda^2 + 10$

$26 = 26\lambda^2$

$\Rightarrow$ $\lambda = 1$ or -1 **[1]**

Put the values of λ in (iii), we get **[1]**

$\vec{r}\,.\,(4\hat{i}+2\hat{j}-4\hat{k})=6$ and $\vec{r}\,.\,(-2\hat{i}+4\hat{j}+4\hat{k})=6$

22. Let A be the event that the machine produces 2 acceptable items.

Also let B_1 represent the event of correct setup and B_2 represent the event of incorrect setup. **[1]**

Now, $P(B_1) = 0.8$, $P(B_2) = 0.2$

$P(A|B_1) = 0.9 \times 0.9$ and $P(A|B_2) = 0.4 \times 0.4$ **[1]**

Therefore, $$P(B_1 \mid A)=\frac{P(B_1)P(A \mid B_1)}{P(B_1)P(A \mid B_1)+P(B_2)P(A \mid B_2)}$$

$$=\frac{0.8\times0.9\times0.9}{0.8\times0.9\times0.9+0.2\times0.4\times0.4}=\frac{648}{680}=0.95$$ **[2]**

Training helps us to operate tools and machines properly thereby reducing product defect and hence improving efficiency, effectiveness and productivity.

23. Let the probability of getting a tail in the biased coin be p.

Therefore, $P(T)=p$

Hence, $P(H)=3p$

We know that, $P(T)+P(H)=1$

i.e. $p+3p=1$

i.e. $4p=1$

i.e. $p=\frac{1}{4}$

Therefore, $P(T)=\frac{1}{4}, P(H)=\frac{3}{4}$

The Sample Space of the experiment of tossing the coin twice is,

$S=\{HH,HT,TH,TT\}$

Let X : Number of Tails

$X=\{0,1,2\}$

$$P(0)=P(H)P(H)=\frac{3}{4}\times\frac{3}{4}=\frac{9}{16}$$

$$P(1)=P(HT)+P(TH)=\frac{3}{4}\times\frac{1}{4}+\frac{1}{4}\times\frac{3}{4}=\frac{6}{16}$$

$$P(2)=P(TT)=\frac{1}{4}\times\frac{1}{4}=\frac{1}{16}$$

X	0	1	2
$P(X)$	$\frac{9}{16}$	$\frac{6}{16}$	$\frac{1}{16}$
$XP(X)$	$\frac{0}{16}$	$\frac{6}{16}$	$\frac{2}{16}$
$X^2P(X)$	$\frac{0}{16}$	$\frac{6}{16}$	$\frac{4}{16}$

[2]

Mathematical Expectation, $E(X)=\sum XP(X)=\frac{0+6+2}{16}=\frac{8}{16}=\frac{1}{2}=0.50$ **[1]**

Also, $E(X^2)=\sum X^2P(X)=\frac{0+6+4}{16}=\frac{10}{16}=\frac{5}{8}$

Now, $VAR(X)=E(X^2)-[E(X)]^2$

$$VAR(X)=\frac{10}{16}-\frac{4}{16}=\frac{6}{16}=\frac{3}{8}$$

$$SD=\sqrt{VAR(X)}=\sqrt{\frac{3}{8}}=\frac{\sqrt{6}}{4}$$ **[1]**

24. $f:(0,\infty)\to \mathbf{R}$

$$f(x)=9x^2+6x-5$$

Let, $x_1,x_2\in(0,\infty), x_1\neq x_2$

Then, $f(x_1)=f(x_2)$

i.e. $9x_1^2+6x_1-5=9x_2^2+6x_2-5$

i.e. $9(x_1^2-x_2^2)+6(x_1-x_2)=0$

i.e. $3(x_1-x_2)[(x_1+x_2)+2]=0$

$\Rightarrow$ $x_1=x_2$ as $x_1,x_2\in(0,\infty)$ therefore $[(x_1+x_2)+2]\neq 0$

$\Rightarrow$ f is one-one **[1+½]**

Now, as $x\in(0,\infty)$

$\Rightarrow x\geq 0 \quad \Rightarrow 6x\geq 0$

$\Rightarrow x^2\geq 0 \quad \Rightarrow 9x^2\geq 0$

Adding the two gives $\Rightarrow 9x^2+6x\geq 0$

$\Rightarrow 9x^2+6x-5\geq -5$

$\Rightarrow f(x)\geq -5$

$\Rightarrow f(x)\in[-5,\infty)$

$\Rightarrow$ f is not ONTO as codomain is **R**. **[1+½]**

$\Rightarrow$ f is not Invertible.

However if the codomain is changed to $[-5, \infty)$ then the function becomes ONTO and therefore Invertible.

For inverse function exchange x with y. Therefore the required inverse function of **[1]**

$y=9x^2+6x-5$ is

i.e. $x=9y^2+6y-5$

i.e. $9y^2 + 6y - (5 + x) = 0$

i.e. $y = \dfrac{-6 \pm \sqrt{6^2 + 4(9)(5 + x)}}{18}$

i.e. $y = \dfrac{-6 \pm \sqrt{36 + 36(5 + x)}}{18}$

i.e. $y = \dfrac{-6 \pm 6\sqrt{1 + (5 + x)}}{18}$

i.e. $y = \dfrac{-1 \pm \sqrt{6 + x}}{3}$ as $y \in (0, \infty)$

i.e. $f^{-1}(y) = \dfrac{-1 + \sqrt{6 + y}}{3}$ **[2]**

OR

Given: $A = (\mathbb{N} \cup \{0\}) \times (\mathbb{N} \cup \{0\})$

$(a,b) * (c,d) = (a + c, b + d)$ for all $(a, b), (c, d) \in A$

(i) For Commutatively:

To Check that: $(a,b) * (c,d) = (c,d) * (a,b)$

$LHS = (a,b) * (c,d) = (a + c, b + d)$

$RHS = (c,d) * (a,b) = (c + a, d + b) = (a + c, b + d)$ **[2]**

Hence, $LHS = RHS$

Hence, * is commutative

(ii) For Associativity;

To Check that: $(a,b) * [(c,d) * (x,y)] = [(a,b) * (c,d)] * (x,y)$

$LHS = (a,b) * [(c,d) * (x,y)] = (a,b) * (c + x, d + y) = (a + c + x, b + d + y)$

$RHS = [(a,b) * (c,d)] * (x,y) = (a + c, b + d) * (x,y) = (a + c + x, b + d + y)$ **[2]**

Hence, $LHS = RHS$

Hence, * is associative

(iii) Let (x, y) be the identity element in A.

Then, $(a,b) * (x,y) = (a,b)$

i.e. $(a + x, b + y) = (a,b)$

$\Rightarrow$ $a + x = a,\ b + y = b$

$\Rightarrow$ $x = 0,\ y = 0$

$\Rightarrow$ $(x,y) = (0,0) \in A$

$\therefore$ identify element is (0, 0) **[2]**

25. $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$

$A = IA$

$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$ **[½]**

$$\begin{bmatrix}1&2&3\\0&1&2\\3&1&1\end{bmatrix}=\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}A$$ By Applying: $R_1 \rightarrow R_2$ **[½]**

$$\begin{bmatrix}1&2&3\\0&1&2\\0&-5&-8\end{bmatrix}=\begin{bmatrix}0&1&0\\1&0&0\\0&-3&1\end{bmatrix}A$$ By Applying: $R_3 \rightarrow 3R_3 - 3R_1$

$$\begin{bmatrix}1&0&-1\\0&1&2\\0&-5&-8\end{bmatrix}=\begin{bmatrix}-2&1&0\\1&0&0\\0&-3&1\end{bmatrix}A$$ By Applying: $R_1 \rightarrow 3R_1 - 2R_2$ **[½]**

$$\begin{bmatrix}1&0&-1\\0&1&2\\0&0&2\end{bmatrix}=\begin{bmatrix}-2&1&0\\1&0&0\\5&-3&1\end{bmatrix}A$$ By Applying: $R_3 \rightarrow R_3 + 5R_2$

$$\begin{bmatrix}1&0&-1\\0&1&2\\0&0&1\end{bmatrix}=\begin{bmatrix}-2&1&0\\1&0&0\\\frac{5}{2}&\frac{-3}{2}&\frac{1}{2}\end{bmatrix}A$$ By Applying: $R_3 \rightarrow \frac{1}{2}R_3$ **[½]**

$$\begin{bmatrix}1&0&0\\0&1&2\\0&0&1\end{bmatrix}=\begin{bmatrix}\frac{1}{2}&-\frac{1}{2}&\frac{1}{2}\\1&0&0\\\frac{5}{2}&\frac{-3}{2}&\frac{1}{2}\end{bmatrix}A$$ By Applying: $R_1 \rightarrow R_1 + R_3$ **[½]**

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}=\begin{bmatrix}\frac{1}{2}&-\frac{1}{2}&\frac{1}{2}\\-4&3&-1\\\frac{5}{2}&\frac{-3}{2}&\frac{1}{2}\end{bmatrix}A$$ By Applying: $R_2 \rightarrow R_2 - 2R_3$ **[½]**

Hence, $A^{-1}=\begin{bmatrix}\frac{1}{2}&-\frac{1}{2}&\frac{1}{2}\\-4&3&-1\\\frac{5}{2}&\frac{-3}{2}&\frac{1}{2}\end{bmatrix}$ By Applying: $R_2 \rightarrow R_2 - 2R_3$ **[1]**

$X = BA^{-1}$

$$=\begin{bmatrix}1&3&-2\end{bmatrix}\begin{bmatrix}\frac{1}{2}&-\frac{1}{2}&\frac{1}{2}\\-4&3&-1\\\frac{5}{2}&\frac{-3}{2}&\frac{1}{2}\end{bmatrix}$$

$$=\begin{bmatrix}\frac{1}{2}-12-5 & -\frac{1}{2}+9+3 & \frac{1}{2}-3-1\end{bmatrix}$$

$$=\begin{bmatrix}-\frac{33}{2} & \frac{23}{2} & \frac{-7}{2}\end{bmatrix}$$

26. Let $I=\int_0^{10}|x-5|\,dx$

(Using $\int_a^b f(x)dx=\int_a^c f(x)dx+\int_c^b f(x)dx$) **[½]**

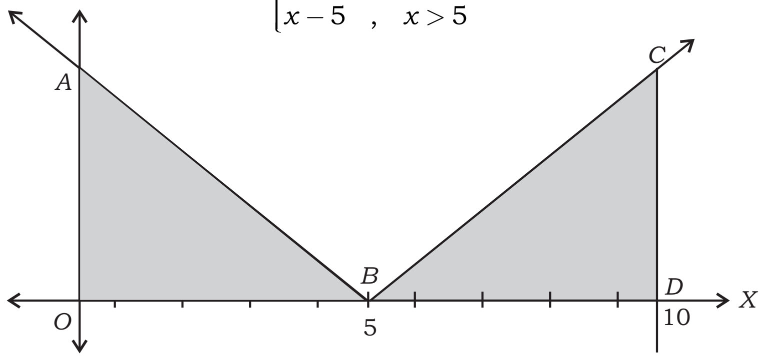
$= -\int_0^5 (x-5)dx + \int_5^{10}(x-5)dx$ **[½]**

$= -\left[\frac{x^2}{2} - 5x\right]_0^5 + \left[\frac{x^2}{2} - 5x\right]_5^{10}$

$= -\left(\frac{25}{2} - 25\right) + (50-50) - \left(\frac{25}{2} - 25\right)$

$= -25 + 50 = 25$ **[½]**

Now, $f(x) = |x-5| = \begin{cases} 5-x & , \quad x < 5 \\ 0 & , \quad x = 5 \\ x-5 & , \quad x > 5 \end{cases}$ **[1]**



[2]

The value of integral I represent area of the region $\Delta OAB + \Delta BCD$.

i.e., Area of shaded region = 25 square units. **[½]**

27. $I = \int_0^{\pi} \frac{xdx}{a^2\cos^2 x + b^2 \sin^2 x}$

$I = \int_0^{\pi} \frac{(\pi - x)dx}{a^2\cos^2(\pi - x) + b^2\sin^2(\pi - x)}$ $\left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$

$I = \pi\int_0^{\pi} \frac{dx}{a^2\cos^2 x + b^2\sin^2 x} - \int_0^{\pi}\frac{xdx}{a^2\cos^2 x + b^2 \sin^2 x}$ **[½]**

$I = \pi\int_0^{\pi} \frac{dx}{a^2\cos^2 x + b^2\sin^2 x} - I$

$2I = \pi\int_0^{\pi} \frac{dx}{a^2\cos^2 x + b^2\sin^2 x}$

$I = \frac{\pi}{2}\int_0^{\pi} \frac{dx}{a^2\cos^2 x + b^2\sin^2 x}$ **[1]**

$I = \frac{\pi}{2} \times 2\int_0^{\frac{\pi}{2}} \frac{dx}{a^2\cos^2 x + b^2\sin^2 x}$ $\left[\because \int_0^{2a} f(x)dx = 2\int_0^a f(x)dx \text{ if } f(2a-x) = f(x)\right]$ **[1]**

$I = \pi\int_0^{\frac{\pi}{2}} \frac{\sec^2 xdx}{a^2 + b^2\tan^2 x}$ **[½]**

Put, $b\tan x = t$ so that $b\sec^2 x\,dx = dt$. Also, when $x = 0, t = 0$ and when $x = \frac{\pi}{2}$, $t \to \infty$.

Therefore, $I = \frac{\pi}{b}\int_0^{\infty}\frac{dt}{a^2+t^2} = \frac{\pi}{b}.\frac{1}{a}\left[\tan^{-1}\frac{t}{a}\right]_0^{\infty} = \frac{\pi}{ab}\left[\frac{\pi}{2} - 0\right] = \frac{\pi^2}{2ab}$ **[3]**

OR

Here, $a = 0$, $b = 1$, $f(x) = 3x^2 + 2x + 1$, $nh = b - a = 1$

$\Rightarrow$ $f(a) = f(0) = 1$

$f(a+h) = f(h) = 3h^2 + 2h + 1$

$f(a+2h) = f(2h) = 3.2^2.h^2 + 2.2h + 1$

$f(a+3h) = f(3h) = 3.3^2h^2 + 2.3h + 1$

$f(a+\overline{n-1}\ h) = f(\overline{n-1}\ h) = 3(n-1)^2h^2 + 2.(n-1)h + 1$ **[2]**

$\therefore \quad \int_0^1 (3x^2+2x+1)\,dx$

$\lim_{x\to 0} h\left[3(h^2)(1^2+2^2+.....+(n-1)^2) + 2h(1+2+...+(n-1)+n\right]$ **[½]**

$\left(\because \quad \int_a^b f(x)\ dx = \lim_{h\to 0} h\left[f(a)+f(a+h)+.....+f(a+\overline{n-1}\ h)\right]\right)$ **[½]**

$= \lim_{h\to 0} h\left[3h^2\frac{n(n-1)(2n-1)}{6} + 2h\frac{n(n-1)}{2} + n\right]$

$= \lim_{h\to 0}\left[\frac{nh(nh-h)(2nh-h)}{2} + \frac{2nh(nh-h)}{2} + nh\right]$

$= \lim_{h\to 0}\left[\frac{(1-h)}{2} + (2-h) + 1\right]$ ...[By using $nh = 1$] **[3]**

$= 1 + 1 + 1 = 3$

$$\left[\begin{array}{l}\therefore \quad 1^2+2^2+....+(n-1)^2 = \frac{n(n-1)(2n-1)}{6} \\ \quad\quad 1+2+....+(n-1) = \frac{n(n-1)}{2}\end{array}\right]$$

28 Two lines $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$ are coplanar if $(\vec{a}_2 - \vec{a}_1)\cdot(\vec{b}_1 \times \vec{b}_2) = 0$.

Here, $\vec{a}_1 = \hat{i} + 2\hat{j} - 4\hat{k}$, $\quad \vec{b}_1 = 2\hat{i} + 3\hat{j} + 6\hat{k}$,

$\vec{a}_2 = 3\hat{i} + 2\hat{j} - 5\hat{k}$ $\quad \vec{b}_2 = -2\hat{i} + 3\hat{j} + 8\hat{k}$

Hence, $(\vec{a}_2 - \vec{a}_1)\cdot(\vec{b}_1 \times \vec{b}_2) = \begin{vmatrix} 3-1 & 2-2 & -5+4 \\ 2 & 3 & 6 \\ -2 & 3 & 8 \end{vmatrix}$

$= \begin{vmatrix} 2 & 0 & -1 \\ 2 & 3 & 6 \\ -2 & 3 & 8 \end{vmatrix}$

$= 2(24-18) + 0(16+12) - (6+6)$

$= 12 - 12$

$= 0$ **[2]**

Hence planes are coplanar.

Now, the equation of the plane containing two lines is given by,

i.e., $(\vec{r} - \vec{a}_1)\cdot(\vec{b}_1 \times \vec{b}_2) = 0$

i.e., $\begin{vmatrix} x-1 & y-2 & z+4 \\ 2 & 3 & 6 \\ -2 & 3 & 8 \end{vmatrix} = 0$

i.e., $(x-1)(24-18) - (y-2)(16+12) + (z+4)(6+6) = 0$

i.e., $6x - 6 - 28y + 56 + 12z + 48 = 0$

i.e., $6x - 28y + 12z + 98 = 0$ **[4]**

29. Let the airline sell x ticket of executive class and y tickets of economy class.

The mathematical formulation of the given problem is as follows.

Maximize $Z = 1000x + 600y$ ...(1)

Subject to be constraints

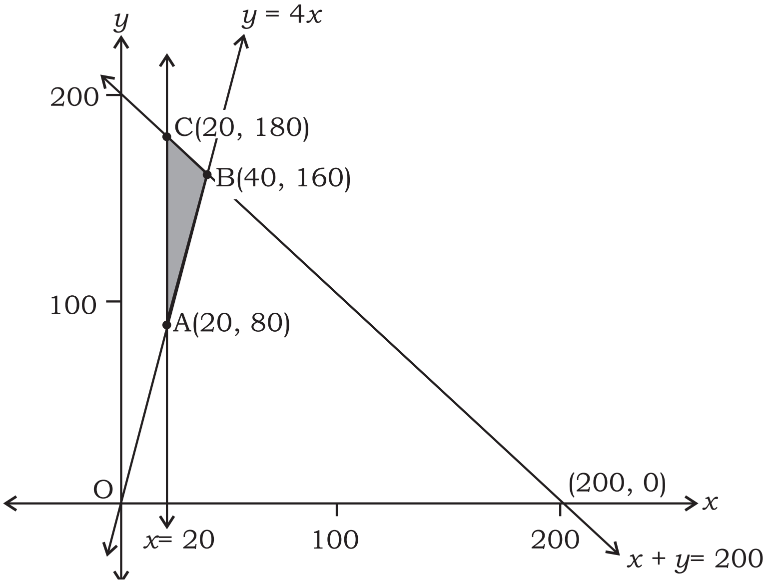
$x + y \leq 200$ ...(2)

$x \geq 20$ ...(3)

$y - 4x \geq 0$ ...(4)

$x, y \geq 0$ ...(5) **[2+½]**

The feasible region determined by the constraints is as follows:



[2+½]

The corner points of the feasible region area $A(20, 80)$, B (40, 160) and C (20, 180).

The values of Z at these corner points are as follows:

Corner point	***Z = 1000x + 600y***
A (20, 80)	68000
B (40, 160)	136000 → Maximum
C (20, 180)	128000

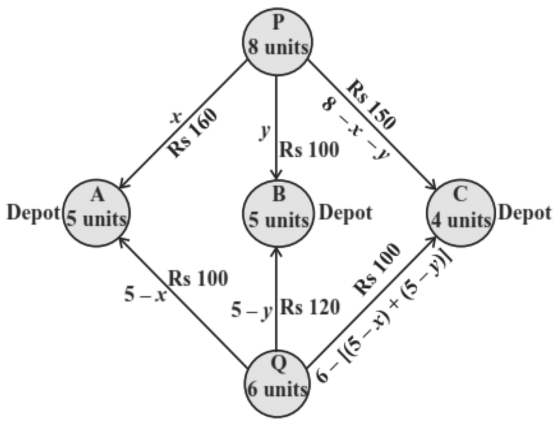
The maximum value of Z is 136000 at (40, 160).

Thus, 40 tickets of executive class and 160 tickets of economy class should be sold to maximize the profit and then maximum profit is Rs. 136000. **[1]**

OR

The problem can be explained diagrammatically as follows:

Let x units and y units of the commodity be transported from the factory at P to the depots at A and B respectively. Then $(8 - x - y)$ units will be transported to depot at C.



Hence, we have $x \geq 0$, $y \geq 0$ and $8 - x - y \geq 0$

i.e. $x \geq 0$, $y \geq 0$ and $x + y \leq 8$

Now, the weekly requirement of the depot at A is 5 units of the commodity. Since x units are transported from the factory at P, the remaining $(5 - x)$ units need to be transported from the factory at Q. Obviously, $5 - x \geq 0$,

i.e. $x \leq 5$.

Similarly, $(5 - y)$ and $6 - (5 - x + 5 - y) = x + y - 4$ units are to be transported from the factory at Q to the depots at B and C respectively.

Thus, $5 - y \geq 0$, $x + y - 4 \geq 0$

i.e. $y \leq 5$, $x + y \geq 4$

Total transportation cost Z is given by

$$Z = 160\,x + 100y + 100(5 - x) + 120(5 - y) + 100(x + y - 4) + 150(8 - x - y) = 10(x - 7\,y + 190)$$

Therefore, the problem reduces to Minimise $Z = 10\,(x - 7y + 190)$ subject to the constraints:

$x \geq 0$, $y \geq 0$ ... (1)

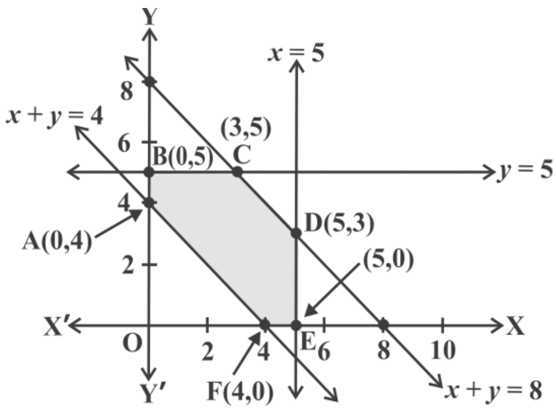
$x + y \leq 8$ ... (2)

$x \leq 5$ ... (3)

$y \leq 5$ ... (4)

and $x + y \geq 4$ ... (5) **[2+½]**

The shaded region $ABCDEF$ represented by the constraints (1) to (5) is the feasible region.



[2+½]

Observe that the feasible region is bounded. The coordinates of the corner points of the feasible region are (0, 4), (0, 5), (3, 5), (5, 3), (5, 0) and (4, 0). Let us evaluate Z at these points.

Corner Point	$Z = 10\,(x - 7\,y + 190)$
(0, 4)	1620
(0, 5)	1550 ← Minimum
(3, 5)	1580
(5, 3)	1740
(5, 0)	1950
(4, 0)	1940

[1]

From the table, we see that the minimum value of Z is 1550 at the point (0, 5).

Hence, the optimal transportation strategy will be to deliver 0, 5 and 3 units from the factory at P and 5, 0 and 1 units from the factory at Q to the depots at A, B and C respectively. Corresponding to this strategy, the transportation cost would be minimum, i.e., Rs 1550.

❖❖❖❖❖